CS 40: Final Examination

Department of Computer Science
University of California, Santa Barbara
Closed-Book, 3 hours

Fall 2006

Instructions

- Before you answer any questions, print your name and perm number.
- Read each question carefully. Make sure that you clearly understand each question before answering it.
- Put your answer to each question on its own page.
- You may wish to work out an answer on scratch paper before writing it on your answer page; answers that are difficult to read may lose points for that reason.
- You may not leave the room during the examination, even to go to the bathroom.
- You may not use any personal devices, such as calculators, PDAs, or cell phones.
1. (6 points) Give a recursive definition of the sequence \( \{a_n\} \), \( n = 1, 2, 3, \ldots \) if \( a_n = n(n + 1) \).

Answer:

By differences,

(a) \( a_n = n(n + 1) = n^2 + n \).
(b) \( a_{n-1} = (n - 1)n = n^2 - n \).
(c) \( a_n - a_{n-1} = 2n \).
(d) \( a_n = a_{n-1} + 2n \).

By ratios,

(a) \( a_n = n(n + 1) \).
(b) \( a_{n-1} = (n - 1)n \).
(c) \( a_n/a_{n-1} = (n + 1)/(n - 1) \).
(d) \( a_n = a_{n-1}(n + 1)/(n - 1) \).

In both cases, an initial condition is \( a_1 = 2 \).

2. (6 points) Devise a recursive algorithm for computing \( n^2 \), where \( n \) is a nonnegative integer using the fact that \( (n + 1)^2 = n^2 + 2n + 1 \).

Answer:

```c
int f( int n )
{
    if ( n == 0 )
        return 0;
    else
        return f( n - 1 ) + 2*n - 1;
}
```

3. (6 points) How many ways are there to pick 2 different cards from a standard 52-card deck such that the 1\textsuperscript{st} card is a spade and the 2\textsuperscript{nd} card is not a Queen?

Answer: Use the sum principle to partition the 2-card sequence into 2 sets of ordered pairs: those whose 1\textsuperscript{st} card is:

(a) the Queen of spades: There is 1 way to pick the Queen of spades, and 48 ways to select the 2\textsuperscript{nd} card.
(b) some other spade: There are 12 non-Queen spades, and for each there are 47 non-Queens cards remaining from which to select the 2\textsuperscript{nd} card.

Thus, the answer is 48 + 12 \cdot 47.
4. (6 points) Let \( d \) be a positive integer. Show that among any group of \( d + 1 \) (not necessarily consecutive) integers there are 2 with exactly the same remainder when they are divided by \( d \).

Answer: Let \( n_i \), for \( i = 1, \ldots, d + 1 \) be the \( d + 1 \) integers.

Let \( r_i \), for \( i = 1, \ldots, d + 1 \) be their respective remainders, when divided by \( d \).

Since there are \( d + 1 \) remainders whose values must be in the interval \([0, d−1]\), there must be 2 remainders of the same value. Their corresponding integers, thus have the same remainder when divided by \( d \).

5. (6 points) How many ways are there to pick a 5-person basketball team from 10 players, if the weakest player and the strongest player must be on the team?

Answer: Since the weakest and strongest players must be on the team, there are 3 team members that can be selected from the remaining 8 players: \( \binom{8}{3} \).

6. (7 points) There are 6 different French books, 8 different Russian books, and 5 different Spanish books. How many ways are there to arrange the books in a row on a shelf with all the books of the same language grouped together?

Answer: There are 3! ways to order the languages on the shelf. For each of these ways, there are:

- 6! ways to order the French books
- 8! ways to order the Russian books
- 5! ways to order the Spanish books.

The total number of arrangements thus is 3!6!8!5!.

7. (7 points) Show that

\[ 1 = \binom{n}{0} 2^n - \binom{n}{1} 2^{n-1} + \binom{n}{2} 2^{n-2} - \cdots + (-1)^n \binom{n}{n} 2^0. \]

Answer: The Binomial Theorem is

\[ (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i. \]

Evaluate the Binomial Theorem at \( x = 2 \) and \( y = -1 \).

8. (7 points) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

Answer: The number is the same as the number of nonnegative integer solutions to the equation

\[ p + n + d + q + h = 20 \]

which is \( \binom{20 + 5 - 1}{5 - 1} \).
9. (7 points) How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

Answer: \[
\frac{20!}{(2!)^4(3!)^3}.
\]

10. (7 points) How many ways are there to distribute 20 different toys among 5 children?

Answer: Abstractly, we are asked to distribute 20 distinct objects into 5 distinct boxes. A distribution is an assignment of each distinct object into 1 of 5 distinct boxes. Since there are 20 objects, the total number of distributions is \(5^{20}\).

11. (7 points) How many ways are there to distribute 40 identical jelly beans among 7 UCSB students, if each student gets at least 1 bean?

Answer: The number is the same as the number of nonnegative integer solutions to the equation

\[x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 40 - 7\]

which is \(\binom{40 - 7 + 7 - 1}{7 - 1}\).

12. (7 points)

(a) Find a recurrence relation for the number of ways to climb \(n\) stairs, if the person climbing the stairs can take 1, 2, or 3 stairs at a time.

(b) What are the initial conditions?

(c) How many ways can this person climb a flight of 8 stairs?

Answer: \(a_n = a_{n-1} + a_{n-2} + a_{n-3}, n \geq 4\), with initial conditions:

- \(a_1 = 1\), namely \((1)\)
- \(a_2 = 2\), namely \((1, 1), (2)\)
- \(a_3 = 4\), namely \((1, 1, 1), (1, 2), (2, 1), (3)\)

\[a_4 = 4 + 2 + 1 = 7.\]
\[a_5 = 7 + 4 + 2 = 13.\]
\[a_6 = 13 + 7 + 4 = 24.\]
\[a_7 = 24 + 13 + 7 = 44.\]
\[a_8 = 44 + 24 + 13 = 81.\]

13. (7 points) Suppose that there are \(n = 2^k\) teams in an elimination tournament, where there are \(n/2\) games in the first round, with the \(n/2 = 2^{k-1}\) winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.

Answer:

Looking at some instances:
\[ r_{2^1} = 1. \]
\[ r_{2^2} = 2 = r_{2^1} + 1. \]
\[ r_{2^3} = 3 = r_{2^2} + 1. \]

From this, the recurrence evidently is \( r_{2^k} = r_{2^{k-1}} + 1 \), or equivalently, \( r_n = r_{n/2} + 1 \) for \( n = 2^k \).

14. (7 points) Find the number of positive integers not exceeding 1,000 that are either the square or the cube of an integer.

Answer:

(a) The set, \( S \), of integer squares not exceeding 1000 is \( S = \{1^2, 2^2, \ldots, 31^2\} \).
(b) The set, \( C \), of integer cubes not exceeding 1000 is \( C = \{1^3, 2^3, \ldots, 10^3\} \).
(c) \( |S \cup C| = |S| + |C| - |S \cap C| \).
(d) \( S \cap C = \{1^2)^3, (2^2)^3, \text{and}(32^3\} \).
(e) Thus, \( |S \cup C| = 31 + 10 - 3 = 38 \).

15. (7 points) How many ways are there to distribute 7 different donuts to 3 different policemen such that each policeman gets at least 1 donut?

Answer:

(a) Let \( A_i \) be the set of distributions such that policeman \( i \) gets 0 donuts.
(b) Then, we want
\[
|A_1 \cap A_2 \cap A_3| = |U| - \left( \binom{3}{1} |A_i| \right) + \left( \binom{3}{2} |A_i \cap A_j| \right) - \left( \binom{3}{3} |A_1 \cap A_2 \cap A_3| \right).
\]
(c)

i. \( |U| = 3^7 \).
ii. \( |A_i| = 2^7 \).
iii. \( |A_i \cap A_j| = 1^7 \).
iv. \( |A_1 \cap A_2 \cap A_3| = 0 \).

The answer thus is \( 3^7 - 3 \cdot 2^7 + 3 \).