

$q$-cube enumerator polynomial of Fibonacci graphs

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Abstract

We consider a $q$-analogue of the cube polynomial of Fibonacci graphs. These bivariate polynomials satisfy a recurrence relation similar to the standard one. They refine the count of the number of hypercubes of a given dimension in Fibonacci graphs by keeping track of the distances of the hypercubes to the all 0 vertex. For $q = 1$ they specialize to the standard cube polynomials.

We also investigate the divisibility properties of the $q$-analogues and show that the quotient polynomials for the appropriate indices have nonnegative integral polynomials in $q$ as coefficients. These results have many corollaries which include expressions involving the $q$-analogues of the Fibonacci numbers themselves and their convolutions as they relate to hypercubes in Fibonacci graphs. Many of our developments can be viewed as refinements of enumerative results given by Klavžar and Mollard [9].

Keywords: Hypercube Fibonacci number Fibonacci graph cube enumerator polynomial $q$-analogue

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1 Introduction

A graph $G = (V, E)$ with vertex set $V$ and edge set $E$ can be used to represent an interconnection network. In this representation, $V$ denotes the processors and $E$ denotes the communication links between processors. The hypercube graph $H_n$ of dimension $n$ is one of the basic model for interconnection networks. The vertices of $H_n$ are represented by all binary strings of length $n$ and two vertices are adjacent if and only if they differ in exactly one position. The graph distance between two vertices of a graph is the length of the shortest path connecting these vertices. In $H_n$, the graph distance between two vertices is given by the Hamming distance between the corresponding binary strings; this is the number of different bits of their binary representations. In $H_n$ the weight of a vertex is defined as the number of ones in the corresponding string, that is, the Hamming weight of the string.

In [8] Fibonacci graphs (also known as Fibonacci cubes) were introduced as a new model of computation for interconnection networks. The Fibonacci graph $\Gamma_n$ of dimension $n$ is a subgraph of $H_n$, where the vertices correspond to those without two consecutive 1s in their string representation. In other words, if we label the vertices of $\Gamma_n$ ($n \geq 1$) by using binary strings $b_1b_2\ldots b_n$ of length $n$, then the vertices of $\Gamma_n$ have the property that $b_ib_{i+1} = 0$ for all $i \in \{1, 2, \ldots, n-1\}$. For convenience $\Gamma_0$ is defined as $H_0$, the graph with a single vertex and no edges.

In literature many interesting properties and applications of the Fibonacci graphs are presented. Their usage as interconnection networks and properties that are important in network design are given in [8, 4]. In [10] the usage in theoretical chemistry and some results on the structure of Fibonacci graphs, including representations, recursive construction, hamiltonicity, the nature of the degree sequence and some enumeration results are presented. The characterization of maximal induced hypercubes in $\Gamma_n$ appears in [11]. Results on disjoint hypercubes in $\Gamma_n$ are presented in [5]. The cube polynomial of $\Gamma_n$ which is the starting point of this paper is studied in [9] and many interesting related results are obtained.

In this paper we consider the $q$-analogue of the cube polynomial of the Fibonacci graphs. Many of our results are extensions of the work of Klavžar and Mollard as our $q$-cube polynomial $c_n(x; q)$ is a refinement of the cube polynomial $c_n(x)$ given in [9]. Furthermore, the $q$-analogue adds a geometric meaning to the polynomials; the $c_n(x; q)$ satisfy a simple recursion similar to the recursion for the cube polynomial $c_n(x)$ and have a combinatorial
interpretation as enumerators of the hypercubes in $\Gamma_n$ in which distance information of each hypercubes to the all 0 vertex is catalogued. For example

$$c_2(x) = 3 + 2x$$

since $\Gamma_2$ contains three $H_0$’s and two $H_1$’s, whereas

$$c_2(x; q) = 1 + 2q + 2x$$

expresses the fact that two of the three $H_0$ are at distance 1 from 00 and the other at distance 0; and both $H_1$’s are at distance 0 from 00 (i.e. they contain 00).

Certain divisibility properties of the cube polynomials were noted in [9]. Our results extend these divisibility properties and also includes information about the nature of the quotient polynomials. Interestingly, the quotients as polynomials in $x$, have coefficients that are polynomials in $q$ which have nonnegative integral coefficients themselves.

The distance information of the hypercubes in $\Gamma_n$ maintained in $c_n(x; q)$ also has an interpretation in terms of the ranks when $\Gamma_n$ is viewed as a subposet of the Boolean algebra $H_n$, but this is not the emphasis of the present work.

The paper is organized as follows: In Section 2 we give some preliminaries. We present our $q$-cube enumerator polynomial in Section 3 and investigate divisibility properties in Section 4. In Section 5 we present additional results including the role of Fibonacci numbers and their $q$-analogues in the construction of $c_n(x; q)$, and a closed form expression for the simple $q$-analogue of the hypercube’s own subcube enumerator.

## 2 Preliminaries

In this section we present some notation and preliminary results related to Fibonacci graphs. We start with the description of a hypercube. An $n$-dimensional hypercube (or $n$-cube) $H_n$ is the simple graph with vertex set

$$V(H_n) = \{v_1v_2\cdots v_n \mid v_i \in \{0, 1\}, \ 1 \leq i \leq n\}.$$ 

The number of vertices in $H_n$ is $2^n$ and the number of these without two consecutive 1s is enumerated by the Fibonacci numbers. From this point of view Fibonacci graph $\Gamma_n$ can be considered as a subgraph of $H_n$, obtained
from $H_n$ by removing all vertices containing consecutive 1s. The number of vertices of the Fibonacci graph $\Gamma_n$ is $F_n$, where $F_0 = 1$, $F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. These are the Fibonacci numbers $f_n$ shifted by 2: i.e. $F_n = f_{n+2}$ where $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. In Figure 1 we present first 6 Fibonacci graphs with their vertices labeled with the corresponding binary strings in the hypercube graph. Since there is

![Figure 1: Fibonacci graphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_5$.](image)

a close relationship between hypercubes and Fibonacci graphs it is natural to consider the number of $k$-dimensional hypercubes in $\Gamma_n$ in more detail. The enumerator of these subcubes in the Fibonacci graph $\Gamma_n$ was considered in [9]. Here we are considering a generalization of these polynomials (see, Section 3).

Consider the $q$-analogue of the Fibonacci numbers given by $f_0(q) = 0$, $f_1(q) = 1$, and

$$f_n(q) = f_{n-1}(q) + q f_{n-2}(q) \quad (1)$$

for $n \geq 2$. This $q$-analogue is simpler than the standard one defined by

$$f_n = f_{n-1} + q^{n-2} f_{n-2} \quad (2)$$

due to Schur, which was studied by Carlitz, Cigler, and others in the literature [1, 2], [3]. These are also different from the Fibonacci polynomials [6] which are defined by

$$F_n(x) = x F_{n-1}(x) + F_{n-2}(x)$$

for $n \geq 2$ with $F_0(x) = 0$ and $F_1(x) = 1$. 

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Using (1), first $f_n(q)$ for $n \geq 2$ are computed as:

1, 1 + $q$, 1 + 2$q$, 1 + 3$q + q^2$, 1 + 4$q + 3q^2$, 1 + 5$q + 6q^2 + q^3$, ...

It is well known that

$$\sum_{n \geq 0} f_n t^n = \frac{t}{1 - t - t^2}.$$  

Similarly one can easily obtain the generating function of $f_n(q)$ as

$$\sum_{n \geq 0} f_n(q) t^n = \frac{t}{1 - t - qt^2}. \quad (3)$$

Let $h_{n,k}$ denote the number of $k$-dimensional hypercubes in the Fibonacci graph $\Gamma_n$. Then the cube polynomial, or the cube enumerator polynomial $c_n(x)$ of $\Gamma_n$ is defined as

$$c_n(x) = \sum_{k \geq 0} h_{n,k} x^k. \quad (4)$$

A few of these cube polynomials are given below:

$$
\begin{align*}
c_0(x) &= 1, \\
c_1(x) &= 2 + x, \\
c_2(x) &= 3 + 2x, \\
c_3(x) &= 5 + 5x + x^2, \\
c_4(x) &= 8 + 10x + 3x^2, \\
c_5(x) &= 13 + 20x + 9x^2 + x^3, \\
c_6(x) &= 21 + 38x + 22x^2 + 4x^3, \\
c_7(x) &= 34 + 71x + 51x^2 + 14x^3 + x^4.
\end{align*}
$$

Evidently the constant terms are the number of $H_0$’s, i.e. the number of vertices of $\Gamma_n$. Therefore

$$c_n(0) = F_n = f_{n+2}.$$  

Many interesting results on $c_n(x)$ and $h_{n,k}$ in (4) are given in [9]. It can be observed that the numbers in Table 1 satisfy the recursion

$$h_{n,k} = h_{n-1,k} + h_{n-2,k} + h_{n-2,k-1}. \quad (5)$$

The first column entries ($k = 0$) of the table are $F_0, F_1, F_2, \ldots$ and the diagonal entries are 1, 1, 0, 0, 0, 0, ... After these, the other entries can be filled row by row by using the recursion (5).

Next we use the $q$-analogue (1) to study generalization of $c_n(x)$. 

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Table 1: The table of coefficients of the cube polynomials $c_n(x)$ by rows. The
entry in row $n$, column $k$ is the coefficient $h_{n,k}$, the number of $k$-dimensional
hypercubes in $\Gamma_n$.

<table>
<thead>
<tr>
<th>$n \backslash k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>1</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
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<td>0</td>
</tr>
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<td>0</td>
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<td>4</td>
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<td>0</td>
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<td>71</td>
<td>51</td>
<td>14</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

3 $q$-cube polynomials of the Fibonacci graphs

In this section we define the polynomials $c_n(x; q)$ of the Fibonacci graph
$\Gamma_n$. They will be defined in terms of the distance of the corresponding $k$-
dimensional hypercubes to the all 0 vertex (our starting vertex, that is, $\Gamma_0$).
Recall that the distance between two subgraphs of a graph is the smallest
graph distance between pairs of vertices taken one from each.

The polynomial $c_n(x; q)$ is defined as the sum of terms of the form $q^d x^k$,
one for each hypercube subgraph of $\Gamma_n$. The exponent $k$ is the dimension of
the hypercube under consideration, and the exponent $d$ is its distance to the
all 0 vertex in $\Gamma_n$.

It is useful to think of $c_n(x; q)$ as a polynomial in $x$ whose coefficients are
polynomials in $q$. First few $c_n(x; q)$ are as follows:

\[
\begin{align*}
c_0(x; q) & = 1, \\
c_1(x; q) & = 1 + q + x, \\
c_2(x; q) & = 1 + 2q + 2x, \\
c_3(x; q) & = 1 + 3q + q^2 + (3 + 2q)x + x^2, \\
c_4(x; q) & = 1 + 4q + 3q^2 + (4 + 6q)x + 3x^2. \\
\end{align*}
\]

Now we illustrate the structure of $c_2(x; q)$ and $c_3(x; q)$ in more detail. Recall
that there are 3 vertices and 2 edges in the graph of $\Gamma_2$ as in Figure 1. The
0-dimensional hypercubes are the vertices of the graph. There is a single
vertex having distance 0 to the vertex 00 (i.e. 00 itself) and there are 2
vertices having distance 1. Therefore the coefficient of $x^0$ in $c_2(x; q)$ is $1 + 2q$. Similarly, 1-dimensional hypercubes are the edges of the graph and there are a total of 2 of those, each having distance zero to the vertex 00. Therefore the coefficient of $x^1$ is 2. This gives $c_2(x; q) = 1 + 2q + 2x$.

Similarly to construct $c_3(x; q)$ we consider all hypercubes in $\Gamma_3$ having dimension $k < 3$ and their distances to the 000. For $k = 0$ we know that there are 5 vertices in the graph giving 0-dimensional hypercubes. The vertex 000 has distance 0, the vertices 010, 100 and 001 each have distance 1 and the vertex 101 has distance 2 to 000. So the coefficient of $x^0$ is $1 + 3q + q^2$.

Now consider $k = 1$, that is, 1-dimensional hypercubes in the graph. We know that they are the edges of the graph and from Figure 2 we see that there are 3 with distance 0 and 2 with distance 1 to the vertex 000. So the coefficient of $x^1$ in $c_3(x; q)$ is $3 + 2q$.

Finally consider $k = 2$. There is only one 2-dimensional hypercube in $\Gamma_3$ and this hypercube contains the vertex 000. So the contribution from 2-dimensional subcubes is $x^2$. Therefore we get $c_3(x; q) = (1 + 3q + q^2) + (3 + 2q)x + x^2$. A graphical presentation of these hypercubes in $\Gamma_3$ and their contribution to $c_3(x, q)$ is presented in Figure 2.

We next determine the generating function for the $q$-cube polynomial $c_n(x; q)$ and relate it to the $q$-analogues of the Fibonacci numbers in (1). Before this result however, we present the following recursion which allows for the calculation of the polynomials and which is central to what follows.

**Lemma 1** For $n \geq 2$ the $q$-cube polynomial $c_n(x; q)$ satisfies

$$c_n(x; q) = c_{n-1}(x; q) + (q + x)c_{n-2}(x; q)$$

with $c_0(x; q) = 1$ and $c_1(x; q) = 1 + q + x$.

**Proof** We see that

$$c_2(x; q) = 1 + 2q + 2x = c_1(x; q) + (q + x)c_0(x; q).$$

Now we can use induction on $n$. Assume that the recursion holds for $n - 1$. We make use of what [10] calls the "fundamental decomposition" for the construction of $\Gamma_n$ from $\Gamma_{n-1}$ and $\Gamma_{n-2}$. $\Gamma_{n-1}$ contains an isomorphic copy of $\Gamma_{n-2}$, say $\Gamma'_{n-2}$, and there are unique edges between the corresponding vertices of the new $\Gamma_{n-2}$ and this copy $\Gamma'_{n-2}$. It follows that there are only three kinds of hypercubes in $\Gamma_n$:
Case 1: A $k$-dimensional hypercube in $\Gamma_{n-1}$ remains a $k$-dimensional hypercube in $\Gamma_n$ and the distances of these cubes to the all 0 vertex remain unchanged. By induction, these are enumerated by $c_{n-1}(x; q)$.

Case 2: Any $k$-dimensional hypercube in $\Gamma_{n-2}$ is again a $k$-dimensional hypercube in $\Gamma_n$, and the distances of these cubes in $\Gamma_n$ to the all 0 vertex go up by 1 due to the edges identifying the corresponding vertices in $\Gamma_{n-2}$ and $\Gamma_{n-2}$. This increase in the distance to the all 0 vertex by 1 means multiplication by $q$, and the contribution of these hypercubes is $q c_{n-2}(x; q)$.

Case 3: A $k$-dimensional hypercube in $\Gamma_{n-2}$ has an isomorphic copy in $\Gamma_{n-2}$ and all the corresponding vertices of these $k$-dimensional hypercubes are connected by unique edges. Therefore these two $k$-dimensional hypercubes together with the edges connecting them gives a $(k + 1)$-dimensional hypercube in $\Gamma_n$. Also note that the distances of these cubes to the all 0 vertex remain unchanged. The contribution of these hypercubes is $x c_{n-2}(x; q)$, since...

$$c_3(x; q) = (1 + 3q + q^2) + (3 + 2q)x + x^2$$
multiplication by $x$ has the effect of increasing the dimension by 1. Adding up these three contributions we obtain recursion (6).

Now we consider the generating function of the $c_n(x; q)$.

**Proposition 1** The generating function of the $q$-cube polynomial $c_n(x; q)$ is

$$
\sum_{n \geq 0} c_n(x; q)t^n = \frac{1 + t(q + x)}{1 - t - t^2(q + x)}.
$$

**Proof** Let $S = \sum_{n \geq 0} c_n(x; q)t^n$. We know that $c_0(x; q) = 1$, $c_1(x; q) = 1 + q + x$ and $c_n(x; q)$ satisfies recursion (6). Therefore $S$ satisfies

$$
S - 1 - t(1 + q + x) = t(S - 1) + t^2(q + x)S,
$$

which gives the desired result. $ullet$

Next we solve the recursion in (6) directly to find $c_n(x; q)$ in explicit form.

**Theorem 1** For any nonnegative integer $n$, the $q$-cube polynomial $c_n(x; q)$ has degree $\left\lfloor \frac{n+1}{2} \right\rfloor$ in $x$ and

$$
c_n(x; q) = \frac{1}{2^{n+1}} \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+2}{2i+1} (1 + 4(q + x))^i.
$$

**Proof** We know that the characteristic equation of the recursion in (6) is

$$
r^2 - r - (q + x) = 0.
$$

This equation gives an explicit expression in the form

$$
c_n(x; q) = \frac{(1 + \theta)^{n+2} - (1 - \theta)^{n+2}}{2^{n+2}\theta}
$$

where $\theta = \sqrt{1 + 4(q + x)}$. Using binomial expansions for $(1 \pm \theta)^{n+2}$ and after some algebraic manipulation, we obtain (7). $ullet$

In particular, writing

$$
c_n(x; q) = \sum_{k \geq 0} h_{n,k}(q)x^k,
$$

we obtain the following expression for the coefficient polynomials $h_{n,k}(q)$. 9
Corollary 1 For any nonnegative integer \( n \), the coefficient polynomials of the \( q \)-cube polynomial \( c_n(x; q) \) are

\[
h_{n,k}(q) = \frac{1}{2^{n+1}} \left( \frac{1}{1 + 4q} \right)^k \sum_{i=k}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{i}{k} (1 + 4q)^i.
\]

A few of the polynomials \( h_{n,k}(q) \) are given in Table 2.

<table>
<thead>
<tr>
<th>( n ) ( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
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<td>0</td>
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<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
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<td>3 + 2q</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1 + 4q + 3q^2</td>
<td>4 + 6q</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1 + 5q + 6q^2 + q^3</td>
<td>5 + 12q + 3q^2</td>
<td>6 + 3q</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1 + 6q + 10q^2 + 4q^3</td>
<td>6 + 20q + 12q^2</td>
<td>10 + 12q</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1 + 7q + 15q^2 + 10q^3 + q^4</td>
<td>7 + 30q + 30q^2 + 4q^3</td>
<td>15 + 30q + 6q^2</td>
<td>10 + 4q</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: The table of coefficients of the \( q \)-cube polynomials \( c_n(x; q) \) by rows. The entry in row \( n \), column \( k \) is the coefficient polynomial \( h_{n,k}(q) \).

Using the properties of convolutions we obtain the following result relating the coefficient polynomials \( h_{n,k}(q) \) of the \( q \)-cube \( c_n(x; q) \) and the \( q \)-analogue of the Fibonacci numbers given in (1).

Proposition 2 The coefficient polynomials \( h_{n,k}(q) \) of the \( q \)-cube enumerator \( c_n(x; q) \) are given by

\[
h_{n,k}(q) = \sum f_{i_0}(q)f_{i_1}(q) \cdots f_{i_k}(q)
\]

where the summation is over all \( i_0, i_1, \ldots, i_k \geq 0 \) with \( i_0 + i_1 + \cdots + i_k = n - k + 2 \).

Proof Recall from Proposition 1 that the generating function of the \( c_n(x; q) \) is

\[
\sum_{n \geq 0} c_n(x; q)t^n = \frac{1 + t(q + x)}{1 - t - t^2(q + x)}.
\]

On the other hand, using (3), the \( (k + 1) \)-fold convolutions of the \( f_n(q) \) have the generating function

\[
\frac{t^{k+1}}{(1 - t - qt^2)^{k+1}}.
\]
Setting
\[ g_k(t; q) = \frac{t^{2k-1}}{(1 - t - qt^2)^{k+1}} \]
for \( k \geq 1 \) with
\[ g_0(t; q) = \frac{t^{-1}}{(1 - t - qt^2)} - \frac{1}{t} \]
and calculating directly, we find
\[
\sum_{k \geq 0} g_k(t; q) x^k = -\frac{1}{t} + \frac{1}{t(1 - t - qt^2)} \sum_{k \geq 0} \left( \frac{xt^2}{1 - t - qt^2} \right)^k
\]
\[= \frac{1 + t(q + x)}{1 - t - t^2(q + x)}.\]
This is the generating function of the \( c_n \) of (10). Therefore the \( g_k(t; q) \) are the generating functions of the columns of Table 2. This proves the proposition by equating the coefficients of \( t^n x^k \) in the two expressions.

Note that using Proposition 2, the polynomials in the first column \((k = 0)\) of Table 2 are given explicitly as
\[
f_{n+2}(q) = \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n - i + 1}{i} q^i.
\]
The second column \((k = 1)\) polynomials are
\[
\sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n - i}{i+1} (i + 1) q^i,
\]
and in general, we have the following expression for the entry in row \( n \), column \( k \):

**Proposition 3** The coefficient polynomials \( h_{n,k}(q) \) of the \( q \)-cube enumerator \( c_n \) of the Fibonacci graph \( \Gamma_n \) are given explicitly by
\[
h_{n,k}(q) = \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n - i + 1 - k}{i + k} \binom{i + k}{k} q^i.
\]
The above proposition can be proved directly from the recurrence in (1), by using induction on \( k \) and verifying a binomial identity.

**Remark**

From the two different expressions for \( h_{n,k}(q) \) in Corollary 1 and Proposition 3, we obtain the following identity for \( n \geq 0 \) and \( k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \):

\[
\frac{1}{2^{n+1}} \left( \frac{4}{1+4q} \right)^k \sum_{i=k}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+2}{2i+1} \binom{4q}{i} (1+4q)^i = \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n-i+1-k}{i+k} \binom{i+k}{k} q^i.
\]

**4 Divisibility properties of the \( q \)-cube polynomials**

In this section we present divisibility properties of the \( q \)-cube polynomials of the Fibonacci graphs. We start by some examples.

The polynomial \( c_1(x; q) = 1 + q + x \) divides \( c_4(x; q) \), \( c_7(x; q) \), \( c_{10}(x; q) \), \ldots since

\[
c_4(x; q) = c_1(x; q) (1 + 3q + 3x),
\]
\[
c_7(x; q) = c_1(x; q) \left( 1 + 6q + 9q^2 + q^3 + (6 + 18q + 3q^2)x + (9 + 3q)x^2 + x^3 \right),
\]
\[
c_{10}(x; q) = c_1(x; q) \left( 1 + 9q + 27q^2 + 29q^3 + 6q^4 + (9 + 54q + 87q^2 + 24q^3)x + (27 + 87q + 36q^2)x^2 + (29 + 24q)x^3 + 6x^4 \right).
\]

Similarly, \( c_2(x) = 1 + 2q + 2x \) divides \( c_6(x; q) \), \( c_{10}(x; q) \), \( c_{14}(x; q) \), \ldots since

\[
c_6(x; q) = c_2(x; q) \left( 1 + 4q + 2q^2 + (4 + 4q)x + 2x^2 \right),
\]
\[
c_{10}(x; q) = c_2(x; q) \left( 1 + 8q + 20q^2 + 16q^3 + 3q^4 + (8 + 40q + 48q^2 + 12q^3)x + (20 + 48q + 18q^2)x^2 + (16 + 12q)x^3 + 3x^4 \right).
\]

Next, \( c_3(x) = 1 + 3q + q^2 + (3 + 2q)x + x^2 \) divides \( c_8(x; q) \), \( c_{13}(x; q) \), \( c_{18}(x; q) \), \ldots since

\[
c_8(x; q) = c_3(x; q) \left( 1 + 5q + 5q^2 + (5 + 10q)x + 5x^2 \right),
\]
\[
c_{13}(x; q) = c_3(x; q) \left( 1 + 10q + 35q^2 + 50q^3 + 25q^4 + q^5 + (10 + 70q + 150q^2 + 100q^3 + 5q^4)x + (35 + 150q + 150q^2 + 10q^3)x^2 + (50 + 100q + 10q^2)x^3 + (25 + 5q)x^4 + x^5 \right).
\]
The above examples hint at certain divisibility properties for the $q$-cube polynomials of the Fibonacci graphs. Also, the coefficients of the polynomials $x^k$ on the right hand side seem to be polynomials in $q$ with nonnegative integral coefficients. These observations are proved in the following theorem.

**Theorem 2** For any $m \geq 0$, $c_m(x; q)$ divides $c_{(m+2)\gamma}(x; q)$ as a polynomial in $x$ for $n \geq 0$. Furthermore, the coefficients of powers of $x$ in the quotient are polynomials in $q$ with nonnegative integral coefficients.

**Proof** To prove the theorem we go back to the expression in (8) and write

$$\frac{c_{(m+2)\gamma} + m(x; q)}{c_{m}(x; q)} = \frac{(1 + \theta)^{m+2} - (1 - \theta)^{m+2}}{2^{m+2}}$$

where $\theta = \sqrt{1 + 4(q + x)}$. For a fixed $m$, denote the quotient on the left of (12) by $\alpha_n(x; q)$. Putting

$$P = (1 + \theta)^{m+2} \quad \text{and} \quad Q = (1 - \theta)^{m+2},$$

we can write

$$\alpha_n(x; q) = \frac{P^{n+1} - Q^{n+1}}{2^{m+2}n(P - Q)}.$$  \hspace{1cm} (14)

For $n = 0, 1$ we obtain

$$\alpha_0(x; q) = 1, \quad \alpha_1(x; q) = \frac{P + Q}{2^{m+2}}.$$  \hspace{1cm} (15)

Writing

$$\alpha_n(x; q) = \left(\frac{P}{P - Q}\right)\left(\frac{P}{2^{m+2}}\right)^n + \left(\frac{-Q}{P - Q}\right)\left(\frac{Q}{2^{m+2}}\right)^n$$

we see that $\alpha_n(x; q)$ is the solution to the second order linear recursion with characteristic equation

$$\left(r - \frac{P}{2^{m+2}}\right)\left(r - \frac{Q}{2^{m+2}}\right) = 0.$$  \hspace{1cm} (16)

This recursion is

$$\alpha_n(x; q) = \left(\frac{P}{2^{m+2}} + \frac{Q}{2^{m+2}}\right)\alpha_{n-1}(x; q) - \left(\frac{PQ}{2^{2m+4}}\right)\alpha_{n-2}(x; q)$$

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for \( n \geq 2 \) with initial values as given in (15). Using (13) and \( \theta = \sqrt{1 + 4(q + x)} \), the recurrence (16) can be written directly in terms of \( x \) as

\[
\alpha_n(x; q) = \alpha_1(x; q)\alpha_{n-1}(x; q) + (-1)^{m+1}(q + x)^{m+2}\alpha_{n-2}(x; q)
\]

(17)

for \( n \geq 2 \). Here \( \alpha_0(x; q) = 1 \) and

\[
\alpha_1(x; q) = \frac{P + Q}{2^{m+2}}
\]

\[
= \frac{1}{2^{m+2}} \left( \left( 1 + \sqrt{1 + 4(q + x)} \right)^{m+2} + \left( 1 - \sqrt{1 + 4(q + x)} \right)^{m+2} \right)
\]

\[
= \frac{1}{2^{m+1}} \sum_{i \geq 0} \frac{m + 2}{2i} (1 + 4(q + x))^i.
\]

Since \( \alpha_1(x; q) \) is a polynomial in \( x \), it follows from the recursion (17) that \( \alpha_n(x; q) \) is a polynomial in \( x \) for all \( n \geq 0 \).

Next we show that the coefficients of \( \alpha_n(x; q) \) are polynomials in \( q \) whose coefficients are nonnegative integers. For integrality, it suffices to show that the coefficients of \( \alpha_1(x; q) \) are integral polynomials and then make use of the recursion (17). To do this, we now take into account the dependence on \( m \) and write \( \alpha_1(x; q) \) as \( \beta_m(x; q) \). Thus

\[
\beta_m(x; q) = \frac{P + Q}{2^{m+2}}
\]

\[
= \frac{1}{2^{m+2}} \left( \left( 1 + \sqrt{1 + 4(q + x)} \right)^{m+2} + \left( 1 - \sqrt{1 + 4(q + x)} \right)^{m+2} \right).
\]

For \( m = 0, 1 \) we have

\[
\beta_0(x; q) = 3 + 2x, \quad \beta_1(x; q) = 4 + 3x.
\]

(18)

Writing

\[
\beta_m(x; q) = \frac{1}{4} \left( 1 + \sqrt{1 + 4(q + x)} \right)^2 \left( 1 + \sqrt{1 + 4(q + x)} \right)^{m}
\]

\[
+ \frac{1}{4} \left( 1 - \sqrt{1 + 4(q + x)} \right)^2 \left( 1 - \sqrt{1 + 4(q + x)} \right)^{m}
\]

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and going through the calculations with the appropriate characteristic equation again, we find that \( \beta_m(x; q) \) is the solution to the second degree linear recurrence equation

\[
\beta_m(x; q) = \beta_{m-1}(x; q) + (q + x)\beta_{m-2}(x; q)
\]

for \( m \geq 2 \) with initial values as \( \beta_0(x; q) \) and \( \beta_1(x; q) \) as given in (18). From this recursion it is evident that the coefficients of \( \alpha_1(x; q) \) are polynomials in \( q \) and their coefficients are nonnegative integers. It is curious that the recurrence satisfied by the \( \beta_m(x; q) \) is the same as the recursion (6) satisfied by the \( q \)-cube polynomials \( c_n(x; q) \), albeit with different initial conditions.

The coefficient of \( x^j \) in \( \alpha_1(x; q) \) is explicitly given by

\[
\frac{1}{2^{m+1}} \sum_{i \geq 0} \binom{m+2}{2i} \binom{i}{j} 4^j (1 + 4q)^{i-j}
\]

and so this is always a polynomial in \( q \) with integer coefficients.

We proved that the coefficients of all of the \( \alpha_n(x; q) \) are polynomials in \( q \) with integer coefficients. The nonnegativity of the coefficients is clear from (17) for \( m \) odd. For \( m \) even, we use the expression (14) for \( \alpha_n(x; q) \). It suffices to show that the coefficients polynomials of the powers of \( x \) have nonnegative coefficients in

\[
\sum_{k=0}^{n} P^k Q^{n-k}.
\]

(19)

For \( 2k < n \), consider the pair

\[
P^k Q^{n-k} + P^{n-k} Q^k = (1 - \theta^2)^{(m+2)k} \left((1 + \theta)^{(m+2)(n-2k)} + (1 - \theta)^{(m+2)(n-2k)}\right)
\]

\[
= (-4(q + x))^{(m+2)k} \sum_{i \geq 0} 2^{i} \binom{m+2}(i-2k)(1 + 4(q + x))^i.
\]

For \( m \) even, the coefficients are nonnegative. In case \( n \) is even, there is a single central term in the sum (19), which is \( P^k Q^k \) for \( k = n/2 \). For \( m \) even, using (13), we obtain that the coefficients of this term are also nonnegative. Therefore the coefficient polynomials in \( \alpha_n(x; q) \) are always sum of polynomials in \( q \) with nonnegative coefficients.

**Remark**

Using (11), the constant term of the quotient can be written as

\[
\frac{c_{(m+2)n+m}(0; q)}{c_m(0; q)} = \frac{f_{(m+2)n+m+2}(q)}{f_{m+2}(q)}.
\]
The integrality of the quotient implies that in particular $f_m(q) | f_{mn}(q)$ for $m, n \geq 1$. If we take $q = 1$ this reduces to the well-known divisibility result [12] of Fibonacci numbers $f_m | f_{mn}$ for $m, n \geq 1$.

**Remark**

Letting $q = 1$, we obtain from Theorem 2 that for any $m \geq 0$, $c_m(x) | c_{(m+2)n+m}(x)$ for $n \geq 0$ which was given in [9, Corollary 6.3, (ii)]. For the numerical case, Theorem 2 also shows that the quotient polynomials have only nonnegative integer coefficients.

## 5 Observations and specializations

For completeness, we present the enumerator for the original hypercubes and some special results on $c_n(x; 1)$.

### 5.1 Hypercube’s own $q$-cube enumerator polynomial

We consider a statistic similar to the one we used for $\Gamma_n$ for the hypercube graph $H_n$ itself. Identify $H_n$ with the Hasse diagram of the poset of all binary strings of length $n$ with $0^n$ at the bottom and $1^n$ at the top and where the covering relation is flipping a 0 to a 1 (This is the lattice of subsets of an $n$-element set.) The rank of a vertex $v = v_1v_2\ldots v_n$ is its Hamming weight. Denote this by $|v|_1$. A $k$-dimensional subcube of $H_n$ can be identified with a $k$-subset of the set of indices $1, 2, \ldots, n$ which is allowed to vary all possible ways. For any $k$-dimensional subcube $H$ of $H_n$, assign the weight

$$w(H) = q^i x^k$$

where $i$ is the smallest rank of the vertices of $H$. This is the same as the distance between $H$ and the all 0 vertex when we consider the Hasse diagram of $H_n$ as a graph. For the Fibonacci graph we have obtained in Theorem 1 that

$$\sum_H w(H) = c_n(x; q) = \frac{1}{2^{n+1}} \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n + 2}{2i + 1} (1 + 4(q + x))^i .$$

The expression for the hypercube $H_n$ itself is simpler.
Proposition 4 For the $n$-dimensional hypercube graph $H_n$, 
\[ \sum_{H} w(H) = (1 + q + x)^n , \]
where the summation is over all subcubes $H$ of $H_n$.

Proof If $H$ is $k$-dimensional, then its vertices are of the form 
\[ v_0 x_1 v_1 x_2 \cdots x_k v_k \]
for $x_1, \ldots, x_k \in \{0, 1\}$ and $v_0 \cdots v_k$ a fixed string of length $n - k$ over $\{0, 1\}$. Clearly the lowest ranked vertex in $H$ has $x_1 = \cdots = x_k = 0$, and the highest one has $x_1 = \cdots = x_k = 1$. Therefore
\[
\sum_{H} w(H) = \sum_{k=0}^{n} x^k \sum_{v_0, \ldots, v_k \in \{0, 1\}^*} q^{\sum_{i} v_i} \sum_{|v_0| + \cdots + |v_k| = n - k} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} n - k \\ i \end{array} \right) q^i = (1 + q + x)^n . \]

5.2 Results for $q = 1$

In this section we present some special results for the case $q = 1$, which is
the case considered in [9]. Besides the known results we also present some
new ones.

The constant term in $c_n(x; 1)$ is the number of vertices $F_n$ of $\Gamma_n$, which
is obtained by taking $x = 0$ in (7). Curiously, this gives
\[
F_n = \frac{1}{2^{n+1}} \sum_{i \geq 0} \left( \begin{array}{c} n + 2 \\ 2i + 1 \end{array} \right) 5^i .
\]
In terms of the Fibonacci numbers $f_n = F_{n-2}$, we obtain the formula

Proposition 5 The Fibonacci numbers $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$ are
given by
\[
f_n = \frac{1}{2^{n-1}} \sum_{i \geq 0} \left( \begin{array}{c} n \\ 2i + 1 \end{array} \right) 5^i .
\]

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Also, from the expression (7) for the $c_n(x; q)$ we immediately get the specialization
\[ c_n\left(-\frac{1 + 4q}{4}; q\right) = \frac{n + 2}{2^n + 1}. \]
This can of course be obtained from the original recursion (6) by setting
\[ x = -\frac{\frac{1 + 4q}{4}}{q} \] and solving the resulting recursion. Also note that the equality
\[ c_n\left(-\frac{5}{4}; 1\right) = \frac{n + 2}{2^n + 1} \] is given in [9, p. 103]. The values $a_n = c_n(1; 1)$ satisfy the recurrence
\[ a_n = a_{n-1} + 2a_{n-2} \]
with $a_0 = 1, a_1 = 3$, giving the Jacobstahl sequence [7]:
\[ 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, \ldots \]

Remark
Evidently, the quotients of the $q$-cube polynomials carry interesting combinatorial information as the coefficients polynomials are all integral and have nonnegative coefficients. It should also be possible to pursue this venue of investigation for Lucas cubes for their $q$-analogues. In another direction, the $q$-cube polynomials themselves can be further refined by making use of the $q$-Fibonacci numbers defined by (2) instead of the ones in (1) that we have used. They would then carry extra information concerning the subcubes of $\Gamma_n$, in terms of their creation history when we look at the repeated fundamental decomposition of each $\Gamma_n$ into $\Gamma_{n-1}$ and $\Gamma_{n-2}$. There are also interesting questions we are considering which are related to the interpretation of the distances of subcubes to the all 0 vertex in $\Gamma_n$ as rank information when the graphs are viewed as partially ordered sets.

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References


