Proofs for Improved Approximation Algorithms for Multimessage Multicasting

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Abstract

This paper has all the proofs for Gonzalez' paper entitled "Improved Approximation Algorithms for Multimessage Multicasting" by T. Gonzalez [1]. One set of proofs are "hand proofs" and the other set is by a Mathematica program. The Mathematica proofs can also be found in http://www.cs.ucsb.edu/~teo.

1 Introduction

In the following sections we include all the proofs for the lemmas in "Improved Approximation Algorithms for Multimessage Multicasting" by T. Gonzalez.

2 Proofs for Lemmas 2.1 - 2.5

2.1 "Hand" Proofs for Lemmas 2.1 - 2.5

These proofs also appear in [1]. They are included here for completeness.

Lemma 2.1 *For the set of values Procedure Coloring is defined $L \leq R$.*

Proof: By applying simple algebraic transformations one can readily see that the definition of L is equivalent to

$$L = (h+1)^2 - \frac{h^2+3h}{2} + \frac{l}{d-1} - \frac{h^2+h-2}{2(d-1)}, \text{ and}$$
$$L = (h+1)^2 + \frac{-(h^2+3h)d+2h+2+2l}{2(d-1)}$$

Substituting $l \leq \frac{h(h+3)d}{2} - h^2$ in the above inequality, and simplifying we obtain

$$L \leq (h+1)^2 + \frac{-2h^2+2h+2}{2(d-1)}$$

Since $l > h$ we multiply the last term of the inequality by $\frac{l-h}{l-h}$ and obtain

$$L \leq (h+1)^2 - \frac{2h^2 l}{2(d-1)(l-h)} + \frac{h(h+1)(h+3)d - 2h^2(h+1)}{2(d-1)(l-h)} + \frac{2h^3 - 2h^2 - 2h}{2(d-1)(l-h)}$$

After simple algebraic manipulations we get

$$L \leq (h+1)^2 - \frac{2h^2 l}{2(d-1)(l-h)} + \frac{(h+1)h(h+3)}{2(l-h)} + \frac{h^3 + h}{2(d-1)(l-h)} = R$$

This completes the proof of the Lemma.

$\square$

Lemma 2.2 *For the set of values Procedure Coloring is defined* $R \geq (h+1)^2 - \frac{h^2}{d-1} + \frac{h+1}{d-1}$.

Proof: Since $l > h$ we multiply the last two terms in left-hand side (LHS) of the inequality by $\frac{2(l-h)}{2(l-h)}$ and obtain

$$LHS = (h+1)^2 - \frac{2lh^2 - 2h^3}{2(d-1)(l-h)} + \frac{(h+1)2(l-h)}{2(d-1)(l-h)}$$

Substituting $l \leq \frac{h(h+3)d}{2} - h^2$ in the rightmost l of the above inequality, we know

$$LHS \leq (h+1)^2 - \frac{2lh^2 - 2h^3}{2(d-1)(l-h)} + \frac{(h+1)(h(h+3)d - 2h^2 - 2h)}{2(d-1)(l-h)}$$

This is equivalent to

$$LHS \leq (h+1)^2 + \frac{h(h+1)(h+3)}{2(l-h)} + \frac{-2lh^2 + h^3 + h}{2(d-1)(l-h)} = R$$

This completes the proof of the Lemma.

$\square$

Lemma 2.3 *The value for* Δ *defined by Procedure Coloration is greater than or equal to* $(h+2)d - 2h$.

Proof: By Lemma 2.1 we partition the proof into the following two cases depending on the relative values of k, L and R.

Case 1: $k \geq R$**.**

Substituting the inequality in Lemma 2.2 in the equation for the conditions of case 1, we know that $k \geq (h+1)^2 - \frac{h^2}{d-1} + \frac{h+1}{d-1}$. The resulting expression after multiplying both sides by $d-1$ and rearranging terms is

$$k(d-1) \geq (h^2 + 2h + 1)d - h(2h+1).$$

Adding $d(h+1) - h$ to both sides and rearranging terms,

$$k(d-1) + h(d-1) + d \geq (h^2 + 3h + 2)d - 2h(h+1).$$

Factoring $h+1$ in the right hand side of the inequality, rearranging terms, and substituting for the value of Δ determined by the procedure, we know

$$\Delta = \tfrac{(k+h+1)d-(k+h)}{h+1} \geq (h+2)d - 2h.$$

This completes the proof for Case 1.

Case 2: $L \leq k < R$.

From the conditions of Case 2, we know that

$$k \geq \tfrac{h^2+h+2}{2} - \tfrac{h^2+h-2}{2(d-1)} + \tfrac{l}{d-1}.$$

This is equivalent to

$$2k(d-1) \geq d(h^2+h+2) - (2h^2+2h) + 2l,$$

which is equivalent to

$$2k(d-1) - (h^2-h-2)d + (2h^2-2h) - 2l \geq 2(h+2)d - 4h.$$

Adding $2(h+2)dl - 4hl$ to both sides of the inequality, distributing terms, and replacing for the value of Δ determined by the procedure we know that

$$\Delta = \frac{((2d-4)h+4d-2)l + 2(d-1)k + (2-d)h^2 + (d-2)h + 2d}{2(l+1)} \geq (h+2)d - 2h.$$

This completes the proof for Case 2 and the Lemma.

$\square$

Lemma 2.4 *At the beginning of the j^{th} iteration of Procedure Coloring each bundle b emanating from processor P_j satisfies $\sum_{i=0}^{h} c_i^b \geq d$.*

Proof: Suppose not. Suppose that there is a bundle b for which $\sum_{i=0}^{h} c_i^b < d$. Let bundle b emanating from processor P_j be a counter-example to the lemma with the least value for $\sum_{i=0}^{h} c_i^b$. In case of ties select the bundle b emanating from P_j with the largest value for c_0^b. If ties persist, then select any bundle with the two properties given above. In what follows we drop the superscript b, since b is fixed.

If $\sum_{i=1}^{h} c_i > 1$, one can easily find another counter-example with smaller $\sum_{i=0}^{h} c_i$, or one with the same $\sum_{i=0}^{h} c_i$ but larger value for c_0. In either case we contradict the properties of our counter-example. So we can assume without loss of generality that $\sum_{i=1}^{h} c_i \leq 1$.

Let $x = \sum_{i=0}^{h} c_i \leq d-1$. If $x=0$ then j is set to zero; otherwise, j is set to the largest value in $[0,h]$ such that $c_j \neq 0$. From Lemma 2.3 we know that $\Delta \geq (h+2)d - 2h$, and since $h \geq 1$ and $d > 1$, we know that $\Delta \geq (h+2)d - 2h > d$. Since the remaining t_{j-1}-forbidden colors must be in $C_{h+1}, C_{h_2}\ldots, C_k$, it then follows that

$$\sum_{i=0}^{k} i \cdot c_i \geq j + (\Delta - x)(h+1).$$

Substituting $x = \sum_{i=0}^{h} c_i \leq d-1$ in the above inequality, we know that

$$\sum_{i=0}^{k} i \cdot c_i \geq j + (\Delta - (d-1))(h+1).$$

By Lemma 2.1 we partition the proof into the following two cases depending on the relative values of k, L and R.

Case 1: $k \geq R$**.**

Substituting the value $\Delta = \frac{d(k+h+1)-(k+h)}{h+1}$ computed by the algorithm for Case 1 in the above inequality

$$\sum_{i=0}^{k} i \cdot c_i \geq j + (\tfrac{d(k+h+1)-(k+h)}{h+1} - (d-1))(h+1).$$

This reduces to

$$\sum_{i=0}^{k} i \cdot c_i \geq j + d(k+h+1) - (k+h) - (d-1)(h+1),$$

which is equivalent to

$$\sum_{i=0}^{k} i \cdot c_i \geq j + k(d-1) + 1.$$

Since $j \geq 0$, we know that $\sum_{i=0}^{k} i \cdot c_i > k(d-1)$. This contradicts the definition of the c_is, and completes the proof for Case 1.

Case 2: $L \leq k < R$**.**

Substituting the value $\Delta = \frac{((2d-4)h+4d-2)l+2(d-1)k+(2-d)h^2+(d-2)h+2d}{2(l+1)}$ computed by the algorithm for Case 2 in the above inequality

$$\sum_{i=0}^{k} ic_i \geq j + \tfrac{k(h+1)(d-1)}{l+1} + \tfrac{2dl(h^2+2h+1)-2l(2h^2+2h)-d(h^3-1)+2(h^3+1)}{2(l+1)}$$

Adding and subtracting $\frac{2(l+1)}{2(l+1)} = 1$, the equation becomes

$$\sum_{i=0}^{k} ic_i \geq j + \tfrac{k(h+1)(d-1)}{l+1} + \tfrac{2dl(h^2+2h+1)-2l(2h^2+2h+1)-d(h^3-1)+2h^3}{2(l+1)} + 1$$

Factoring from the third term $2(d-1)(l-h)$, the expression becomes

$$\sum_{i=0}^{k} ic_i \geq j + \tfrac{k(h+1)(d-1)}{l+1} + ((h+1)^2 + \tfrac{d(h^3+4h^2+3h)-2lh^2+2(-2h^2-h)}{2(d-1)(l-h)})\tfrac{(d-1)(l-h)}{l+1} + 1$$

Factoring $(d-1)$ from the forth term, the expression becomes

$$\sum_{i=0}^{k} ic_i \geq j + \tfrac{k(h+1)(d-1)}{l+1} + ((h+1)^2 + \tfrac{h(h+1)(h+3)}{2(l-h)} + \tfrac{-2lh^2+h^3+h)}{2(d-1)(l-h)})\tfrac{(d-1)(l-h)}{l+1} + 1$$

By the conditions of the case we know that $k < R = (h+1)^2 + \frac{h(h+1)(h+3)}{2(l-h)} + \frac{-2lh^2+h^3+h)}{2(d-1)(l-h)}$. Substituting in the above inequality and simplifying,

$$\sum_{i=0}^{k} i \cdot c_i > j + \frac{k(h+1)(d-1)}{l+1} + \frac{k(d-1)(l-h)}{l+1} + 1,$$

which is equivalent to

$$\sum_{i=0}^{k} i \cdot c_i > j + k(d-1) + 1.$$

Since $j \geq 0$, we know that $\sum_{i=0}^{k} i \cdot c_i > k(d-1)$. This contradicts the definition of the c_is, and completes the proof for Case 2 and the Lemma.

□

Lemma 2.5 *At the beginning of the j^{th} iteration of Procedure Coloring each bundle b emanating from processor P_j satisfies at least one of the inequalities eq.(j), for $0 \leq j \leq h+1$, holds.*

Proof:
Suppose not. Suppose that there is a bundle b for which all the above inequalities do not hold, i.e., $c_0^b < d$; $\sum_{i=0}^{j} c_i^b < (j+2)d - 2j$ for all $1 \leq j \leq h$; and $\sum_{i=0}^{l} c_i^b < (h+2)d - 2h$. Let bundle b emanating from processor P_j be a counter-example to the lemma with the least value for $\sum_{i=0}^{l} c_i$. In case of ties select a bundle b emanating from P_j with the least value for $\sum_{i=0}^{l} c_i$ that is largest in lexicographic order. I.e., c_0^b has the largest value; in case of ties, c_1^b is the largest value; in case of ties, c_2^b is the largest value; in case of ties, and so on. Since b is fixed, in what follows we drop the superscript b.

We define the vector $(d_0, d_1, \ldots, d_k)$ as follows:

$$d_0 = d-1, d_1 = 2d-2, d_2 = d-2, d_3 = d-2, d_4 = d-2, \ldots, d_h = d-2, d_{h+1} = 0,$$
$$d_{h+1} = 0, d_{h+2} = 0, \ldots, d_l = 0,$$
$$d_{l+1} = \Delta - ((h+2)d - 2h - 1), d_{l+2} = 0, \ldots, \text{ and } d_k = 0.$$

Note that by Lemma 2.3, we know that $d_{l+1} \geq 1$. The subvector $(c_1, c_2, \ldots, c_h)$ is either the subvector $(d_1, d_2, \ldots, d_h)$, or it is such that $c_i = d_i$ for all $0 \leq i \leq f < h$, $c_{f+1} < d_{f+1}$ and $c_{f+2}, \ldots c_h$ are zero except possibly for one c_j such that $j \in [f+2 \leq j \leq h]$. Note that if this is not the case then either the problem instance satisfies one of the inequalities, or it does not satisfy any of the inequalities but either it is not one with least $\sum_{i=0}^{l} c_i$ or it is one with least $\sum_{i=0}^{l} c_i$, but not the largest in lexicographic order. In all cases we contradict the properties of the bundle b selected.

By definition, $\sum_{i=0}^{k} i \cdot c_i \leq k(d-1)$. We now establish a contradiction by showing that $\sum_{i=0}^{k} i \cdot c_i > k(d-1)$. It is simple to see that $\sum_{i=0}^{k} i \cdot c_i \geq \sum_{i=0}^{k} i \cdot d_i$. From the definition of the D vector, and $\Delta \geq (h+2)d - 2h$ (Lemma 2.3), we know that

$$\sum_{i=0}^{k} i \cdot d_i = 2d - 2 + \frac{(d-2)(h^2+h-2)}{2} + (\Delta - ((h+2)d - 2h - 1))(l+1).$$

By Lemma 2.1 we partition the proof into the following two cases depending on the relative values of k, L and R.

Case 1: $R \leq k$**.**

Substituting the value for Δ and reordering terms the expression becomes,

$$\sum_{i=0}^{k} i \cdot d_i = \frac{2k(d-1)(l+1) - 2ld(h+1)^2 + dh(h^2-1) + 2l(2h^2+2h+1) - 2h^3}{2(h+1)}.$$

Adding and subtracting 2(h+1) to the numerator

$$\sum_{i=0}^{k} i \cdot d_i = \frac{2k(d-1)(l+1) - 2ld(h+1)^2 + dh(h^2-1) + 2l(2h^2+2h+1) + 2(-h^3+h+1) + 2(h+1)}{2(h+1)}.$$

Multiplying the right hand side by $(\frac{(d-1)(l-h)}{h+1})(\frac{h+1}{(d-1)(l-h)}) = 1$ we know that,

$$\sum_{i=0}^{k} i \cdot d_i = \frac{(d-1)(l-h)}{h+1}(\frac{k(l+1)}{(l-h)} + \frac{-2ld(h+1)^2 + dh(h^2-1) + 2l(2h^2+2h+1) - 2h^3}{2(d-1)(l-h)} + \frac{h+1}{(d-1)(l-h)})$$

Substituting the conditions for Case 1 ($k \geq R = \frac{2ld(h+1)^2 - dh(h^2-1) - 2l(2h^2+2h+1) + 2h^3}{2(d-1)(l-h)}$) in the above equation, we know that

$$\sum_{i=0}^{k} i \cdot d_i \geq \frac{(d-1)(l-h)}{h+1}(\frac{k(l+1)}{(l-h)} - k + \frac{h+1}{(d-1)(l-h)}).$$

Which is equivalent to

$$\sum_{i=0}^{k} i \cdot d_i \geq \frac{(d-1)(l-h)}{h+1}(\frac{k(h+1)}{(l-h)} + \frac{h+1}{(d-1)(l-h)}),$$

and thus

$$\sum_{i=0}^{k} i \cdot d_i \geq k(d-1) + 1.$$

Therefore, $\sum_{i=0}^{k} i \cdot c_i \geq \sum_{i=0}^{k} i \cdot d_i > k(d-1)$. A contradiction. This completes the proof for Case 1.

Case 2: $L \leq k < R$**.**

Substituting the value for Δ and reordering terms the expression becomes,

$$\sum_{i=0}^{k} i \cdot d_i = 2d - 2 + \frac{(d-2)(h^2+h-2)}{2} + \frac{2(d-1)k + (2-d)h^2 + (2-d)h - 2d + 2}{2}.$$

Which is equivalent to

$$\sum_{i=0}^{k} i \cdot d_i = k(d-1) + 1.$$

Therefore, $\sum_{i=0}^{k} i \cdot c_i \geq \sum_{i=0}^{k} i \cdot d_i > k(d-1)$. A contradiction. This completes the proof for Case 1.2 and Case 1.

$\square$

2.2 "Mathematica" Proofs for Lemmas 2.1 - 2.2, and Parts of Lemmas 2.3 - 2.5

Lemma 2.1: For the set of values Procedure Coloring is defined $L \leq R$.

Lemma 2.2: For the set of values Procedure Coloring is defined $R \geq (h+1)^2 - \frac{h^2}{d-1} + \frac{h+1}{d-1}$.

Lemma 2.3: The value for Δ defined by Procedure Coloration is greater than or equal to $(h+2)d - 2h$.

Remaining Part of Lemma 2.4: $\sum_{i=0}^{k} i \cdot c_i \geq j + (\Delta - (d-1))(h+1) > k(d-1)$.

Remaining Part of Lemma 2.5:

$$\sum_{i=0}^{k} i \cdot d_i = 2d - 2 + \frac{(d-2)(h^2+h-2)}{2} + (\Delta - ((h+2)d - 2h - 1))(l+1) > k(d-1).$$

"Mathematica" Program To Prove These Lemmas:

The Mathematica programs use two macros: Simplifying (to simplify expressions) and Subst (to replace one symbol by an expression in another expression). These macros print errors when invalid operations are attempted, and will also print conditions under which the operations performed are valid ones. It is simple to verify that all these conditions hold for the values for which procedure Coloring is defined [1].

The proofs are of the following form. We start with an expression and transform it to another expression, such that the first expression holds if and only if the second expression holds. We keep on performing these operations until we get an expression that is trivially true.

The programs are available in http://www.cs.ucsb.edu/~teo. For formatting purposes some blank spaces we deleted form the programs listed here.

```
Off[General::spell]

"**************************************"

"    MACROS                            "

"**************************************"

simplifying[exp_] := Module[{u,v,vp,lhs,rhs},
  Print[" "];
  Print["SIMPLIFING THE EXPRESSION"];
  Print[" "];
  Print["        ",exp];
  lhs=First[exp];
  rhs=Last[exp];
  u=Together[lhs-rhs];
  Print[" "];
 Print["------> USER MUST VERIFY THAT CONDITION [ ",Denominator[u]," > 0 ] ","HOLDS"];
```

```
  Print[" "];
  Print["THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS "];
  Print[" "];
  v=Numerator[u];
  If[TrueQ[Part[exp,0][v,0]],
     Print["[ ",v," ",Part[exp,0]," ",0," ] THAT EVALUATES TO ",Part[exp,0][ v,0]],
     Part[exp,0][v,0]]

                    ]


subst[exp1_,exp2_,note_] := Module[{u,v,vn,vnn,vp,lhs,rhs},
  Print[" "];
  Print["SUBSTITUTING THE EXPRESSION ‘‘",note,"’’"];
  Print[" "];
  Print["        ",exp1];
  Print[" "];
  Print["IN"];
  Print[" "];
  Print["        ",exp2];
  Print[" "];
  If[TrueQ[(Part[exp1,0]==Part[exp2,0] || Part[exp1,0]==Equal ||
           (Part[exp1,0]==Greater  && Part[exp2,0]==GreaterEqual) ||
           (Part[exp1,0]==GreaterEqual  && Part[exp2,0]==Greater) ||
           (Part[exp1,0]==Less  && Part[exp2,0]==LessEqual) ||
           (Part[exp1,0]==LessEqual  && Part[exp2,0]==Less))],
          Print["------> VALID SUBSTITUTION"],
          Print["------> INVALID SUBSTITUTION"]];
  lhs=First[exp2];
  rhs=Last[exp2];
  u=Together[lhs-rhs];
  Print[" "];
  If[TrueQ[Part[exp2,0]==Equal],
    Print["------> NO ADDITIONAL CONDITION IMPOSED"],
    Print["------> USER MUST VERIFY THAT CONDITION [ ",Denominator[u]," > 0 ] ","HOLDS"]
  v=Numerator[u];
  vp=D[v,First[exp1]];
  Print[" "];
  Print["THE RESULTING EXPRESSION IS "];
  Print[" "];
  vn = v /. First[exp1]->Last[exp1];
  vnn = Simplify[vn];
  If[TrueQ[Part[exp2,0][vnn,0]],
   Print[" [ ",vnn," ",Part[exp2,0]," ",0," ] THAT EVALUATES TO ",Part[exp2,0][vnn,0]],
   Part[exp2,0][vnn,0]]
```

```
                                ]

"************************************"

                   Date[]

"************************************"
Print["Definitions"]
"************************************"

Print["The expression for R is "]

R     =  (h+1)^2 + ((h+1)(h^2+3*h))/(2*(l-h)) + (-2*l*h^2+h^3+h)/(2*(d-1)(l-h))

Print["The expression for L is "]

L     =  (h^2+h+2)/2 +l/(d-1) -(h^2+h-2)/(2*(d-1))

Print["dRestriction := The restriction for d  is "]

dRestriction =  d >= (2*l+2*h^2)/(h*(h+3))

Print["lRestriction := The restriction for l is "]

lRestriction =  l <= (h*(h+3)*d)/2-h^2

Print["RLowerBound := A lower bound for R is "]

RLowerBound        = (h+1)^2 -(h^2)/(d-1)+(h+1)/(d-1)


"************************************"
Print["Lemma 2.1: If dRestriction then L <= R."]
"************************************"

Print["Prove that L <= R."]
Print["Since l > h, we prove instead that L*(l-h) >= R*(l-h)."]
Print["The reason for this is to circumvent Matematica's transformation"]
Print["of an expression with factor (l-h) into one with factor (-l+h)."]
Print["The expression is "]
                        L*(l-h)<=R*(l-h)
                        simplifying[%]
                        subst[lRestriction,%,"lRestriction"]
Print["Therefore, L <= R."]
Print["This completes the proof of Lemma 2.1. "]
```

```
"************************************"
Print["Lemma 2.2: If dRestriction then R >= RLowerBound."]
"************************************"

Print["Prove that R >= RLowerBound."]
Print["Since l > h, we prove instead that R*(l-h) >= RLowerBound*(l-h)."]
Print["The reason for this is to circumvent Matematica's transformation"]
Print["of an expression with factor (l-h) into one with factor (-l+h)."]
Print["The expression is "]
                       R*(l-h) >= RLowerBound*(l-h)
                       simplifying[%]
                       subst[dRestriction,%,"dRestriction"]
Print["Therefore, R >= RLowerBound."]
Print["This completes the proof of Lemma 2.2. "]

"************************************"
Print["Definitions"]
"************************************"

Print["The ColorLowerBound is "]
ColorLowerBound            = (h+2)*d -2*h
Print["NumColorsRX := Delta for R <= k is "]
NumColorsRX          = (d*(k+h+1)-(k+h))/(h+1)
Print["NumColorsLR := Delta for L <= k < R is "]
NumColorsLR          = (((2*d-4)*h+4*d-2)*l+2*(d-1)*k+(2-d)*h^2+(d-2)*h+2*d)/(2(l+1))

"************************************"
Print["Lemma 2.3: If dRestriction then Delta >= ColorLowerBound."]
"************************************"

Print["Prove that Delta >= ColorLowerBound. "]
Print["Case 1: R <= k."]
Print["We need to prove NumColorsRX >= ColorLowerBound. The expression is "]
                       NumColorsRX>=ColorLowerBound
                       simplifying[%]
Print["Since RLowerBound <= R and R <= k, then k >= RLowerBound."]
                       subst[k>=RLowerBound,%%,"k >= RLowerBound"]
Print["Therefore, Delta >= ColorLowerBound for Case 1"]
Print["Case 2: L <= k < R."]
Print["We need to prove NumColorsLR >= ColorLowerBound. The expression is "]
                       NumColorsLR >= ColorLowerBound
                       simplifying[%]
Print["By the conditions of the case L <= k."]
                       subst[k>=L,%%,"k >= L"]
```

```
Print["Therefore, Delta >= ColorLowerBound for Case 2"]
Print["This completes the proof of Lemma 2.3."]


"************************************"
Print["For Lemma 2.4 prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1 "]
"************************************"

Print["Case 1: R <= k."]
Print["We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1."]
Print["The expression is "]
          j+(h+1)(delta-(d-1))>=(j+k(d-1)+1)
          simplifying[%]
          subst[delta==NumColorsRX,%,"delta == NumColorsRX"]
Print["Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 1"]

Print["Case 2: L <= k < R."]
Print["We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1."]
Print["First we prove that"]
Print["j+(h+1)(delta-(d-1)) >= j+(k(h+1)(d-1))/(l+1)+R((d-1)(l-h))/(l+1)+1."]
Print["The expression is "]
      j+(h+1)(delta-(d-1))>=j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1
      subst[delta==NumColorsLR,%,"delta == NumColorsLR"]
Print["Therefore,"]
Print["j+(h+1)(delta-(d-1))>=j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1."]
Print["To complete the proof we show that"]
Print["j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1>=j+k(d-1)+1."]
Print["Since R > k, the proof reduces to showing that "]
         j+(k*(h+1)*(d-1))/(l+1)+k*((d-1)*(l-h))/(l+1)+1>=j+k(d-1)+1
         simplifying[%]
Print["Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 2."]
Print["This completes the proof of Lemma 2.4."]


"************************************"
Print["For Lemma 2.5 prove that"]
Print["2d-2+((d-2)(h^2+h-2))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1."]
"************************************"

Print["Case 1: R <= k."]
Print["Prove that"]
Print["2d-2+((d-2)(h^2+h-2))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1."]
Print["The expression is "]
      2d-2+((d-2)*(h^2+h-2))/2+(delta-((h+2)d-2*h-1))*(l+1) >= k(d-1)+1
      simplifying[%]
```

```
        subst[delta==NumColorsRX,%,"delta == NumColorRX"]
        subst[k>=R,%,"k >= R"]
Print["Therefore, the expression holds for Case 1"]
Print["Case 2: L <= k < R."]
Print["Prove that"]
Print["2d-2+((d-2)(h^2+h-2))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1."]
Print["The expression is "]
        2d-2+((d-2)*(h^2+h-2))/2+(delta-((h+2)d-2*h-1))*(l+1) >= k(d-1)+1
        simplifying[%]
        subst[delta==NumColorsLR,%,"delta == NumColorLR"]
Print["Therefore, the expression holds for Case 2"]
Print["This completes the proof of Lemma 2.5"]

Quit
```

Output of "Mathematica" Program:

The symbol & in column 1 indicates continuation from previous "line".

```
Mathematica 2.2 for HP9000 Series 700
Copyright 1988-94 Wolfram Research, Inc.
 -- Terminal graphics initialized --





        **************************************



            MACROS



        **************************************








        **************************************
```

{1996, 7, 15, 17, 17, 48}

****

Definitions

****

The expression for R is

$$(1 + h)^2 + \frac{(1 + h)\,(3\,h + h^2)}{2\,(-h + l)} + \frac{h + h^3 - 2\,h^2\,l}{2\,(-1 + d)\,(-h + l)}$$

The expression for L is

$$\frac{-(-2 + h + h^2)}{2\,(-1 + d)} + \frac{2 + h + h^2}{2} + \frac{l}{-1 + d}$$

dRestriction := The restriction for d is

$$d >= \frac{2\,h^2 + 2\,l}{h\,(3 + h)}$$

lRestriction := The restriction for l is

```
       2   d h (3 + h)
l <= -h  + -----------
                2
```

RLowerBound := A lower bound for R is

```
     2
    h       1 + h           2
-(------) + ------ + (1 + h)
  -1 + d    -1 + d
```

```
**************************************
```

Lemma 2.1: If dRestriction then L <= R.

```
**************************************
```

Prove that L <= R.

Since l > h, we prove instead that L*(l-h) >= R*(l-h).

The reason for this is to circumvent Matematica's transformation

of an expression with factor (l-h) into one with factor (-l+h).

The expression is

```
                      2             2
          -(-2 + h + h )   2 + h + h      1
(-h + l) (-------------- + ---------- + ------) <=
            2 (-1 + d)         2        -1 + d

                                 2           3      2
             2   (1 + h) (3 h + h )     h + h  - 2 h  l
```

```
&    (-h + 1) ((1 + h)  + ------------------ + -------------------)
                              2 (-h + 1)       2 (-1 + d) (-h + 1)
```

SIMPLIFING THE EXPRESSION

```
                              2             2
                  -(-2 + h + h )   2 + h + h      l
        (-h + 1) (-------------- + ---------- + ------) <=
                    2 (-1 + d)         2        -1 + d

                                          2            3      2
                      2   (1 + h) (3 h + h )     h + h  - 2 h  l
&    (-h + 1) ((1 + h)  + ------------------ + -------------------)
                              2 (-h + 1)       2 (-1 + d) (-h + 1)
```

------> USER MUST VERIFY THAT CONDITION [2 (-1 + d) > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

```
                     2      2                      2        2        2
         -3 d h + 2 h  - d h  + 2 l - 3 d h l + 2 h  l - d h  l + 2 l  <= 0
```

SUBSTITUTING THE EXPRESSION ‘‘lRestriction’’

```
               2   d h (3 + h)
        l <= -h  + -----------
                        2
```

IN

```
                    2      2                      2        2        2
        -3 d h + 2 h  - d h  + 2 l - 3 d h l + 2 h  l - d h  l + 2 l  <= 0
```

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 LessEqual 0] THAT EVALUATES TO True

Therefore, L <= R.

This completes the proof of Lemma 2.1.

****

Lemma 2.2: If dRestriction then R >= RLowerBound.

****

Prove that R >= RLowerBound.

Since l > h, we prove instead that R*(l-h) >= RLowerBound*(l-h).

The reason for this is to circumvent Matematica's transformation

of an expression with factor (l-h) into one with factor (-l+h).

The expression is

$$(-h + l) \left((1 + h)^2 + \frac{(1 + h) (3 h + h^2)}{2 (-h + l)} + \frac{h + h^3 - 2 h^2 l}{2 (-1 + d) (-h + l)}\right) >=$$

$$\& \quad \left(-\left(\frac{h^2}{-1 + d}\right) + \frac{1 + h}{-1 + d} + (1 + h)^2\right) (-h + l)$$

SIMPLIFING THE EXPRESSION

$$(-h + l) \left((1 + h)^2 + \frac{(1 + h) (3 h + h^2)}{2 (-h + l)} + \frac{h + h^3 - 2 h^2 l}{2 (-1 + d) (-h + l)}\right) >=$$

$$\& \quad \left(-\left(\frac{h^2}{}\right) + \frac{1 + h}{} + (1 + h)^2\right) (-h + l)$$

```
       -1 + d    -1 + d
```

```
------> USER MUST VERIFY THAT CONDITION [ 2 (-1 + d) > 0 ] HOLDS
```

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

```
                  2         2      3      3
         3 d h - 2 h  + 4 d h  - 2 h  + d h  - 2 l - 2 h l >= 0
```

SUBSTITUTING THE EXPRESSION ‘‘dRestriction’’

```
              2
           2 h  + 2 l
       d >= ----------
           h (3 + h)
```

IN

```
                 2         2      3      3
        3 d h - 2 h  + 4 d h  - 2 h  + d h  - 2 l - 2 h l >= 0
```

```
------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS
```

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, R >= RLowerBound.

This completes the proof of Lemma 2.2.

```
**************************************

Definitions

**************************************
```

```
The ColorLowerBound is


-2 h + d (2 + h)

NumColorsRX := Delta for R <= k is


-h - k + d (1 + h + k)
----------------------
        1 + h

NumColorsLR := Delta for L <= k < R is


                                   2
(2 d + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k +

&      (-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l))



*************************************

Lemma 2.3: If dRestriction then Delta >= ColorLowerBound.


*************************************


Prove that Delta >= ColorLowerBound.

Case 1: R <= k.

We need to prove NumColorsRX >= ColorLowerBound. The expression is


-h - k + d (1 + h + k)
---------------------- >= -2 h + d (2 + h)
        1 + h
```

SIMPLIFING THE EXPRESSION

```
-h - k + d (1 + h + k)
```

```
        --------------------- >= -2 h + d (2 + h)
                1 + h
```

------> USER MUST VERIFY THAT CONDITION [1 + h > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

```
                          2      2
         -d + h - 2 d h + 2 h  - d h  - k + d k >= 0
```

Since RLowerBound <= R and R <= k, then k >= RLowerBound.

SUBSTITUTING THE EXPRESSION ‘‘k >= RLowerBound’’

```
                 2
                h        1 + h           2
        k >= -(------) + ------ + (1 + h)
               -1 + d    -1 + d
```

IN

```
                         2      2
        -d + h - 2 d h + 2 h  - d h  - k + d k >= 0
```

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, Delta >= ColorLowerBound for Case 1

Case 2: L <= k < R.

We need to prove NumColorsLR >= ColorLowerBound. The expression is

```
                                              2
         (2 d + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k +

&       (-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l)) >= -2 h + d (2 + h)
```

SIMPLIFING THE EXPRESSION

$$(2\ d + (-2 + d)\ h + (2 - d)\ h^{2} + 2\ (-1 + d)\ k +$$

$$\&\qquad (-2 + 4\ d + (-4 + 2\ d)\ h)\ l)\ /\ (2\ (1 + l)) >= -2\ h + d\ (2 + h)$$

------> USER MUST VERIFY THAT CONDITION [2 (1 + l) > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

$$-2\ d + 2\ h - d\ h + 2\ h^{2} - d\ h^{2} - 2\ k + 2\ d\ k - 2\ l >= 0$$

By the conditions of the case L <= k.

SUBSTITUTING THE EXPRESSION ‘‘k >= L’’

$$k >= \frac{-(-2 + h + h^{2})}{2\ (-1 + d)} + \frac{2 + h + h^{2}}{2} + \frac{l}{-1 + d}$$

IN

$$-2\ d + 2\ h - d\ h + 2\ h^{2} - d\ h^{2} - 2\ k + 2\ d\ k - 2\ l >= 0$$

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, Delta >= ColorLowerBound for Case 2

This completes the proof of Lemma 2.3.

```
***************************************

For Lemma 2.4 prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1

***************************************


Case 1: R <= k.

We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1.

The expression is


(1 - d + delta) (1 + h) + j >= 1 + j + (-1 + d) k


SIMPLIFING THE EXPRESSION

        (1 - d + delta) (1 + h) + j >= 1 + j + (-1 + d) k

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS


         -d + delta + h - d h + delta h + k - d k >= 0


SUBSTITUTING THE EXPRESSION ''delta == NumColorsRX''

                -h - k + d (1 + h + k)
        delta == ----------------------
                        1 + h

IN

        -d + delta + h - d h + delta h + k - d k >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS
```

```
THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

         Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 1


         Case 2: L <= k < R.

         We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1.

         First we prove that

         j+(h+1)(delta-(d-1)) >= j+(k(h+1)(d-1))/(l+1)+R((d-1)(l-h))/(l+1)+1.

         The expression is


         (1 - d + delta) (1 + h) + j >=

             (-1 + d) (1 + h) k
&    1 + j + ------------------ +
                   1 + l

                                                  2            3      2
                                       2   (1 + h) (3 h + h )     h + h  - 2 h  l
      (-1 + d) (-h + l) ((1 + h)  + ------------------ + -------------------)
                                         2 (-h + l)       2 (-1 + d) (-h + l)
&     ----------------------------------------------------------------------
                                        1 + l


SUBSTITUTING THE EXPRESSION ``delta == NumColorsLR''

                                                       2
        delta == (2 d + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k +

&       (-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l))

IN

        (1 - d + delta) (1 + h) + j >=

             (-1 + d) (1 + h) k
```

```
&    1 + j + ------------------ +
                  1 + l

                                                        2            3      2
                                  2   (1 + h) (3 h + h )     h + h  - 2 h  l
      (-1 + d) (-h + 1) ((1 + h)  + ------------------ + -------------------)
                                       2 (-h + 1)      2 (-1 + d) (-h + 1)
&     ----------------------------------------------------------------------
                                       1 + l
```

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [2 (1 + l) > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore,

j+(h+1)(delta-(d-1))>=j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1.

To complete the proof we show that

j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1>=j+k(d-1)+1.

Since R > k, the proof reduces to showing that

```
         (-1 + d) (1 + h) k   (-1 + d) k (-h + l)
1 + j + ------------------ + ------------------- >=
              1 + l                 1 + l

&    1 + j + (-1 + d) k
```

SIMPLIFING THE EXPRESSION

```
         (-1 + d) (1 + h) k   (-1 + d) k (-h + l)
1 + j + ------------------ + ------------------- >= 1 + j + (-1 + d) k
              1 + l                 1 + l
```

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 2.

This completes the proof of Lemma 2.4.

****

For Lemma 2.5 prove that

2d-2+((d-2)(h^2+h-2))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1.

****

Case 1: R <= k.

Prove that

2d-2+((d-2)(h^2+h-2))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1.

The expression is

```
                                     2
                (-2 + d) (-2 + h + h )
     -2 + 2 d + ---------------------- +
                          2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k
```

SIMPLIFING THE EXPRESSION

```
                                     2
                (-2 + d) (-2 + h + h )
     -2 + 2 d + ---------------------- +
                          2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k
```

```
------> USER MUST VERIFY THAT CONDITION [ 2 > 0 ] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS


                                          2      2
          -2 d + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l -

&     4 d l + 2 delta l + 4 h l - 2 d h l >= 0


SUBSTITUTING THE EXPRESSION ``delta == NumColorRX''

                 -h - k + d (1 + h + k)
        delta == ----------------------
                         1 + h

IN

                                        2      2
        -2 d + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l -

&     4 d l + 2 delta l + 4 h l - 2 d h l >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS

THE RESULTING EXPRESSION IS


                       3      3
          (-(d h) - 2 h  + d h  + 2 h k - 2 d h k + 2 l - 2 d l + 4 h l -

                     2           2
&       4 d h l + 4 h  l - 2 d h  l - 2 k l + 2 d k l) / (1 + h) >= 0


SUBSTITUTING THE EXPRESSION ``k >= R''

                                        2            3      2
                     2   (1 + h) (3 h + h )      h + h  - 2 h  l
        k >= (1 + h)  + ------------------ + -------------------
                            2 (-h + l)       2 (-1 + d) (-h + l)
```

```
IN

                    3      3
        (-(d h) - 2 h  + d h  + 2 h k - 2 d h k + 2 l - 2 d l + 4 h l -

                    2          2
&       4 d h l + 4 h  l - 2 d h  l - 2 k l + 2 d k l) / (1 + h) >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 + h > 0 ] HOLDS

THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

          Therefore, the expression holds for Case 1

          Case 2: L <= k < R.

          Prove that

          2d-2+((d-2)(h^2+h-2))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1.

          The expression is


                                        2
                   (-2 + d) (-2 + h + h )
          -2 + 2 d + ---------------------- +
                             2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k


SIMPLIFING THE EXPRESSION

                                      2
                 (-2 + d) (-2 + h + h )
        -2 + 2 d + ---------------------- +
                           2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k
```

```
------> USER MUST VERIFY THAT CONDITION [ 2 > 0 ] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

                                                 2      2
          -2 d + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l -
&     4 d l + 2 delta l + 4 h l - 2 d h l >= 0

SUBSTITUTING THE EXPRESSION ``delta == NumColorLR''

                                                    2
        delta == (2 d + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k +
&       (-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l))

IN

                                               2      2
        -2 d + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l -
&     4 d l + 2 delta l + 4 h l - 2 d h l >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS

THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

          Therefore, the expression holds for Case 2

          This completes the proof of Lemma 2.5
```

3 Proofs for Lemmas 3.1 - 3.5

3.1 "Hand" Proofs for Lemmas 3.1 - 3.5

These proofs also appear in [1]. They are included here for completeness.

Lemma 3.1 *For the set of values Procedure MColoring is defined $L \leq R$.*

Proof: By applying simple algebraic transformations one can readily see that the definition of L is equivalent to

$$L = (h+1)^2 - \frac{h^2+3h-2}{2} + \frac{l}{d-1} - \frac{h^2+h}{2(d-1)}, \text{ and}$$
$$L = (h+1)^2 + \frac{-(h^2+3h-2)d+2h-2+2l}{2(d-1)}$$

Substituting $l \leq \frac{(h^2+3h-2)d}{2} - h^2 + 2$ in the above inequality, and simplifying we obtain

$$L \leq (h+1)^2 + \frac{-2h^2+2h+2}{2(d-1)}$$

Since $l > h$ we multiply the last term of the inequality by $\frac{l-h}{l-h}$ and obtain

$$L \leq (h+1)^2 - \frac{2h^2 l}{2(d-1)(l-h)} + \frac{(h+1)(h^2+3h-2)d-(h+1)(2h^2-4)}{2(d-1)(l-h)} + \frac{2h^3-2h^2-2h}{2(d-1)(l-h)}$$

After simple algebraic manipulations we get

$$L \leq (h+1)^2 - \frac{2h^2 l}{2(d-1)(l-h)} + \frac{(h+1)(h^2+3h+2)}{2(l-h)} + \frac{h^3+3h+2}{2(d-1)(l-h)} = R$$

This completes the proof of the Lemma.

□

Lemma 3.2 *For the set of values Procedure MColoring is defined $R \geq (h+1)^2 - \frac{h^2}{d-1} + \frac{h+1}{d-1}$*

Proof: Since $l > h$ we multiply the last two terms in left-hand side (LHS) of the inequality by $\frac{2(l-h)}{2(l-h)}$ and obtain

$$LHS = (h+1)^2 - \frac{2lh^2-2h^3}{2(d-1)(l-h)} + \frac{(h+1)2(l-h)}{2(d-1)(l-h)}$$

Substituting $l \leq \frac{(h^2+3h-2)d}{2} - h^2 + 2$ in the rightmost l of the above inequality, we know

$$LHS \leq (h+1)^2 - \frac{2lh^2-2h^3}{2(d-1)(l-h)} + \frac{(h+1)((h^2+3h-2)d-2h^2+4-2h)}{2(d-1)(l-h)}$$

This is equivalent to

$$LHS \leq (h+1)^2 + \frac{(h+1)(h^2+3h-2)}{2(l-h)} + \frac{-2lh^2+h^3+3h+2}{2(d-1)(l-h)} = R$$

This completes the proof of the Lemma.

□

Lemma 3.3 *The value for Δ defined by Procedure MColoring is greater than or equal to $(h+2)d-2h$.*

Proof: By Lemma 3.1 we partition the proof into the following two cases depending on the relative values of k, L and R.

Case 1: $k \geq R$.

Substituting the inequality in Lemma 3.2 in the equation for the conditions of case 1, we know that $k \geq (h+1)^2 - \frac{h^2}{d-1} + \frac{h+1}{d-1}$. The resulting expression after multiplying both sides by $d-1$ and rearranging terms is

$$k(d-1) \geq (h^2+2h+1)d - h(2h+1).$$

Adding $d(h+1)-h$ to both sides and rearranging terms,

$$k(d-1) + h(d-1) + d \geq (h^2+3h+2)d - 2h(h+1).$$

Factoring $h+1$ in the right hand side of the inequality, rearranging terms, and substituting for the value of Δ determined by the procedure, we know

$$\Delta = \frac{(k+h+1)d-(k+h)}{h+1} \geq (h+2)d-2h.$$

This completes the proof for Case 1.

Case 2: $L \leq k < R$.

From the conditions of Case 2, we know that

$$k \geq \frac{h^2+h+4}{2} - \frac{h^2+h}{2(d-1)} + \frac{l}{d-1}.$$

This is equivalent to

$$2k(d-1) \geq d(h^2+h+4) - (2h^2+2h+4) + 2l,$$

which is equivalent to

$$2k(d-1) - (h^2+h)d + (2h^2-2h+4) - 2l \geq 2(h+2)d - 4h.$$

Adding $2(h+2)dl - 4hl$ to both sides of the inequality, distributing terms, and replacing for the value of Δ determined by the procedure we know that

$$\Delta = \frac{((2d-4)h+4d-2)l+2(d-1)k+(2-d)h^2+(d-2)h+4}{2(l+1)} \geq (h+2)d-2h.$$

This completes the proof for Case 2 and the Lemma.

$\square$

Lemma 3.4 *At the beginning of the j^{th} iteration of Procedure MColoring each bundle b emanating from processor P_j satisfies $\sum_{i=0}^{h} c_i^b \geq d$.*

Proof: Suppose not. Suppose that there is a bundle b for which $\sum_{i=0}^{h} c_i^b < d$. Let bundle b emanating from processor q_b be a counter-example to the lemma with the least value for $\sum_{i=0}^{h} c_i^b$. In case of ties select the bundle b emanating from P_j with the largest value for c_0^b. If ties persist, then select any bundle with the two properties given above. In what follows we drop the superscript b, since b is fixed.

If $\sum_{i=1}^{h} c_i > 1$, one can easily find another counter-example with smaller $\sum_{i=0}^{h} c_i$, or one with the same $\sum_{i=0}^{h} c_i$ but larger value for c_0. In either case we contradict the properties of our counter-example. So we can assume without loss of generality that $\sum_{i=1}^{h} c_i \leq 1$.

Let $x = \sum_{i=0}^{h} c_i \leq d-1$. If $x = 0$ then j is set to zero; otherwise, j is set to the largest value in $[0, h]$ such that $c_j \neq 0$. From Lemma 3.3 we know that $\Delta \geq (h+2)d - 2h$, and since $h > 1$ and $d > 1$, we know that $\Delta \geq (h+2)d - 2h > d$. Since the remaining t_{j-1}-forbidden colors must be in $C_{h+1}, C_{h_2} \ldots, C_k$, it then follows that

$$\textstyle\sum_{i=0}^{k} i \cdot c_i \geq j + (\Delta - x)(h+1).$$

Substituting $x = \sum_{i=0}^{h} c_i \leq d-1$ in the above inequality, we know that

$$\textstyle\sum_{i=0}^{k} i \cdot c_i \geq j + (\Delta - (d-1))(h+1).$$

By Lemma 3.1 we partition the proof into the following two cases depending on the relative values of k, L and R.

Case 1: $k \geq R$.

Substituting the value $\Delta = \frac{d(k+h+1)-(k+h)}{h+1}$ computed by the algorithm for Case 1 in the above inequality

$$\textstyle\sum_{i=0}^{k} i \cdot c_i \geq j + (\frac{d(k+h+1)-(k+h)}{h+1} - (d-1))(h+1).$$

This reduces to

$$\textstyle\sum_{i=0}^{k} i \cdot c_i \geq j + d(k+h+1) - (k+h) - (d-1)(h+1),$$

which is equivalent to

$$\sum_{i=0}^{k} i \cdot c_i \geq j + k(d-1) + 1.$$

Since $j \geq 0$, we know that $\sum_{i=0}^{k} i \cdot c_i > k(d-1)$. This contradicts the definition of the c_is, and completes the proof for Case 1.

Case 2: $L \leq k < R$**.**

Substituting the value $\Delta = \frac{((2d-4)h+4d-2)l+2(d-1)k+(2-d)h^2+(d-2)h+4}{2(l+1)}$ computed by the algorithm for Case 2 in the above inequality

$$\sum_{i=0}^{k} c_i \geq j + \frac{k(h+1)(d-1)}{l+1} + \frac{2dl(h^2+2h+1)-2l(2h^2+2h)-d(h^3+h+2)+2(h^3+2h+3)}{2(l+1)}$$

Adding and subtracting $\frac{2(l+1)}{2(l+1)} = 1$, the equation becomes

$$\sum_{i=0}^{k} c_i \geq j + \frac{k(h+1)(d-1)}{l+1} + \frac{2dl(h^2+2h+1)-2l(2h^2+2h+1)-d(h^3+h+2)+2(h^3+2h+2)}{2(l+1)} + 1$$

Factoring from the third term $2(d-1)(l-h)$, the expression becomes

$$\sum_{i=0}^{k} c_i \geq j + \frac{k(h+1)(d-1)}{l+1} + ((h+1)^2 + \frac{d(h^3+4h^2+h-2)-2lh^2+2(-2h^2+h+2)}{2(d-1)(l-h)})\frac{(d-1)(l-h)}{l+1} + 1$$

Factoring $(d-1)$ from the forth term, the expression becomes

$$\sum_{i=0}^{k} c_i \geq j + \frac{k(h+1)(d-1)}{l+1} + ((h+1)^2 + \frac{(h+1)(h^2+3h-2)}{2(l-h)} + \frac{-2lh^2+h^3+3h+2)}{2(d-1)(l-h)})\frac{(d-1)(l-h)}{l+1} + 1$$

By the conditions of the case we know that $k < R = (h+1)^2 + \frac{(h+1)(h^2+3h-2)}{2(l-h)} + \frac{-2lh^2+h^3+3h+2)}{2(d-1)(l-h)}$. Substituting in the above inequality and simplifying,

$$\sum_{i=0}^{k} i \cdot c_i > j + \frac{k(h+1)(d-1)}{l+1} + \frac{k(d-1)(l-h)}{l+1} + 1,$$

which is equivalent to

$$\sum_{i=0}^{k} i \cdot c_i > j + k(d-1) + 1.$$

Since $j \geq 0$, we know that $\sum_{i=0}^{k} i \cdot c_i > k(d-1)$. This contradicts the definition of the c_is, and completes the proof for Case 2 and the Lemma.

□

Lemma 3.5 *If* $d \geq \frac{2l+2h^2-4}{h^2+3h-2}$, $L \leq k$, *and* $h \geq 2$ *then at the beginning of the* j^{th} *iteration of Procedure MColoring each bundle* b *emanating from processor* P_j *satisfies at least one of the inequalities eq.*(j), *for* $0 \leq j \leq h+1$, *holds.*

Proof: Suppose not. Suppose that there is a bundle b for which all the above inequalities do not hold, i.e., $c_0^b < d$; $c_0^b + c_1^b < 2d$; $\sum_{i=0}^{j} c_i^b < (j+2)d - 2j$ for all $2 \leq j \leq h$; and $\sum_{i=0}^{l} c_i^b < (h+2)d - 2h$. Let bundle b emanating from processor P_j be a counter-example to the lemma with the least value for $\sum_{i=0}^{l} c_i$. In case of ties select a bundle b emanating from P_j with the least value for $\sum_{i=0}^{l} c_i$ that is largest in lexicographic order. I.e., c_0^b has the largest value; in case of ties, c_1^b is the largest value; in case of ties, c_2^b is the largest value; in case of ties, and so on. Since b is fixed, in what follows we drop the superscript b.

We define the vector $(d_0, d_1, \ldots, d_k)$ as follows:

$$d_0 = d-1, d_1 = d, d_2 = 2d-4, d_3 = d-2, d_4 = d-2, \ldots, d_h = d-2, d_{h+1} = 0,$$
$$d_{h+1} = 0, d_{h+2} = 0, \ldots, d_l = 0,$$
$$d_{l+1} = \Delta - ((h+2)d - 2h - 1), d_{l+2} = 0, \ldots, \text{ and } d_k = 0.$$

Note that by Lemma 3.3, we know that $d_{l+1} \geq 1$. The subvector $(c_1, c_2, \ldots, c_h)$ is either the subvector $(d_1, d_2, \ldots, d_h)$, or it is such that $c_i = d_i$ for all $0 \leq i \leq f < h$, $c_{f+1} < d_{f+1}$ and $c_{f+2}, \ldots c_h$ are zero except possibly for one c_j such that $j \in [f+2 \leq j \leq h]$. Note that if this is not the case then either the problem instance satisfies one of the inequalities, or it does not satisfy any of the inequalities but either it is not one with least $\sum_{i=0}^{l} c_i$ or it is one with least $\sum_{i=0}^{l} c_i$, but not the largest in lexicographic order. In all cases we contradict the properties of the bundle b selected.

By definition, $\sum_{i=0}^{k} i \cdot c_i \leq k(d-1)$. We now establish a contradiction by showing that $\sum_{i=0}^{k} i \cdot c_i > k(d-1)$. It is simple to see that $\sum_{i=0}^{k} i \cdot c_i \geq \sum_{i=0}^{k} i \cdot d_i$. From the definition of the D vector, and $\Delta \geq (h+2)d - 2h$ (Lemma 3.3), we know that

$$\sum_{i=0}^{k} i \cdot d_i = d + 2(2d-4) + \tfrac{(d-2)(h^2+h-6)}{2} + (\Delta - ((h+2)d - 2h - 1))(l+1).$$

By Lemma 3.1 we partition the proof into the following two cases depending on the relative values of k, L and R.

Case 1: $R \leq k$**.**

Substituting the value for Δ and reordering terms the expression becomes,

$$\sum_{i=0}^{k} i \cdot d_i = \tfrac{2k(d-1)(l+1) - 2ld(h+1)^2 + d(h+1)(h^2-h+2) + 2l(2h^2+2h+1) - 2(h^3+h+1)}{2(h+1)}.$$

Adding and subtracting 2(h+1) to the numerator

$$\sum_{i=0}^{k} i \cdot d_i = \tfrac{2k(d-1)(l+1) - 2ld(h+1)^2 + d(h+1)(h^2-h+2) + 2l(2h^2+2h+1) - 2(h^3+2h+2) + 2(h+1)}{2(h+1)}.$$

Multiplying the right hand side by $(\frac{(d-1)(l-h)}{h+1})(\frac{h+1}{(d-1)(l-h)}) = 1$ we know that,

$$\sum_{i=0}^{k} i \cdot d_i = \tfrac{(d-1)(l-h)}{h+1}\left(\tfrac{k(l+1)}{(l-h)} + \tfrac{-2ld(h+1)^2 + d(h+1)(h^2-h+2) + 2l(2h^2+2h+1) - 2(h^3+2h+2)}{2(d-1)(l-h)} + \tfrac{h+1}{(d-1)(l-h)}\right)$$

Substituting the conditions for Case 1 ($k \geq \frac{2ld(h+1)^2 - d(h+1)(h^2-h+2) - 2l(2h^2+2h+1) + 2(h^3+2h+2)}{2(d-1)(l-h)}$) in the above equation, we know that

$$\sum_{i=0}^{k} i \cdot d_i \geq \frac{(d-1)(l-h)}{h+1}\left(\frac{k(l+1)}{(l-h)} - k + \frac{h+1}{(d-1)(l-h)}\right).$$

Which is equivalent to

$$\sum_{i=0}^{k} i \cdot d_i \geq \frac{(d-1)(l-h)}{h+1}\left(\frac{k(h+1)}{(l-h)} + \frac{h+1}{(d-1)(l-h)}\right),$$

and thus

$$\sum_{i=0}^{k} i \cdot d_i \geq k(d-1) + 1.$$

Therefore, $\sum_{i=0}^{k} i \cdot c_i \geq \sum_{i=0}^{k} i \cdot d_i > k(d-1)$. A contradiction. This completes the proof for Case 1.

Case 2: $L \leq k < R$.

Substituting the value for Δ and reordering terms the expression becomes,

$$\sum_{i=0}^{k} i \cdot d_i = 5d - 8 + \frac{(d-2)(h^2+h-6)}{2} + \frac{2(d-1)k+(2-d)h^2+(2-d)h-4d+6}{2}.$$

Which is equivalent to

$$\sum_{i=0}^{k} i \cdot d_i = k(d-1) + 1.$$

Therefore, $\sum_{i=0}^{k} i \cdot c_i \geq \sum_{i=0}^{k} i \cdot d_i > k(d-1)$. A contradiction. This completes the proof for Case 2 and the Lemma.

□

3.2 "Mathematica" Proofs for Lemmas 3.1 - 3.2, and Parts of Lemmas 3.3 - 3.5

Lemma 3.1: For the set of values Procedure MColoring is defined $L \leq R$.

Lemma 3.2: For the set of values Procedure MColoring is defined $R \geq (h+1)^2 - \frac{h^2}{d-1} + \frac{h+1}{d-1}$

Lemma 3.3: The value for Δ defined by Procedure MColoring is greater than or equal to $(h+2)d - 2h$.

Remaining Part of Lemma 3.4: $\sum_{i=0}^{k} i \cdot c_i \geq j + (\Delta - (d-1))(h+1) > k(d-1)$.

Remaining Part of Lemma 3.5:

$$\sum_{i=0}^{k} i \cdot d_i = 2d - 2 + \frac{(d-2)(h^2+h-2)}{2} + (\Delta - ((h+2)d - 2h - 1))(l+1) > k(d-1).$$

"Mathematica" Program For The Proofs of These Lemmas:

The Mathematica programs use two macros: Simplifying (to simplify expressions) and Subst (to replace one symbol by an expression in another expression). These macros print errors when invalid operations are attempted, and will also print conditions under which the operations performed are valid ones. It is simple to verify that all these conditions hold for the values for which procedure MColoring is defined [1].

The proofs are of the following form. We start with an expression and transform it to another expression, such that the first expression holds if and only if the second expression holds. We keep on performing these operations until we get an expression that is trivially true.

The programs are available in http://www.cs.ucsb.edu/~teo. For formatting purposes some blank spaces we deleted form the programs listed here.

```
Off[General::spell]

"**************************************"

"    MACROS                            "

"**************************************"

simplifying[exp_] := Module[{u,v,vp,lhs,rhs},
  Print[" "];
  Print["SIMPLIFING THE EXPRESSION"];
  Print[" "];
  Print["        ",exp];
  lhs=First[exp];
  rhs=Last[exp];
  u=Together[lhs-rhs];
  Print[" "];
 Print["------> USER MUST VERIFY THAT CONDITION [ ",Denominator[u]," > 0 ] ","HOLDS"];
  Print[" "];
  Print["THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS "];
  Print[" "];
  v=Numerator[u];
  If[TrueQ[Part[exp,0][v,0]],
     Print["[ ",v," ",Part[exp,0]," ",0," ] THAT EVALUATES TO ",Part[exp,0][ v,0]],
     Part[exp,0][v,0]]

                     ]


subst[exp1_,exp2_,note_] := Module[{u,v,vn,vnn,vp,lhs,rhs},
  Print[" "];
  Print["SUBSTITUTING THE EXPRESSION ‘‘",note,"’’"];
  Print[" "];
```

```
  Print["        ",exp1];
  Print[" "];
  Print["IN"];
  Print[" "];
  Print["        ",exp2];
  Print[" "];
  If[TrueQ[(Part[exp1,0]==Part[exp2,0] || Part[exp1,0]==Equal ||
           (Part[exp1,0]==Greater  && Part[exp2,0]==GreaterEqual) ||
           (Part[exp1,0]==GreaterEqual  && Part[exp2,0]==Greater) ||
           (Part[exp1,0]==Less  && Part[exp2,0]==LessEqual) ||
           (Part[exp1,0]==LessEqual  && Part[exp2,0]==Less))],
          Print["------> VALID SUBSTITUTION"],
          Print["------> INVALID SUBSTITUTION"]];
  lhs=First[exp2];
  rhs=Last[exp2];
  u=Together[lhs-rhs];
  Print[" "];
  If[TrueQ[Part[exp2,0]==Equal],
    Print["------> NO ADDITIONAL CONDITION IMPOSED"],
    Print["------> USER MUST VERIFY THAT CONDITION [ ",Denominator[u]," > 0 ] ","HOLDS"]
  v=Numerator[u];
  vp=D[v,First[exp1]];
  Print[" "];
  Print["THE RESULTING EXPRESSION IS "];
  Print[" "];
  vn = v /. First[exp1]->Last[exp1];
  vnn = Simplify[vn];
  If[TrueQ[Part[exp2,0][vnn,0]],
   Print[" [ ",vnn," ",Part[exp2,0]," ",0," ] THAT EVALUATES TO ",Part[exp2,0][vnn,0]],
   Part[exp2,0][vnn,0]]
                              ]

"***************************************"

                    Date[]

"***************************************"
Print["Definitions"]
"***************************************"

Print["The expression for R is "]

R     =  (h+1)^2 + ((h+1)(h^2+3*h-2))/(2*(l-h)) + (-2*l*h^2+h^3+3*h+2)/(2*(d-1)(l-h))

Print["The expression for L is "]
```

```
L     =  (h^2+h+4)/2 +l/(d-1) -(h^2+h)/(2*(d-1))

Print["dRestriction := The restriction for d  is "]

dRestriction =  d >= (2*l+2*h^2-4)/(h^2+3*h-2)

Print["lRestriction := The restriction for l is "]

lRestriction =  l <= ((h^2+3*h-2)*d)/2-h^2+2

Print["RLowerBound := A lower bound for R is "]

RLowerBound        = (h+1)^2 -(h^2)/(d-1)+(h+1)/(d-1)


"***********************************"
Print["Lemma 3.1: If dRestriction then L <= R."]
"***********************************"

Print["Prove that L <= R."]
Print["Since l > h, we prove instead that L*(l-h) >= R*(l-h)."]
Print["The reason for this is to circumvent Matematica's transformation"]
Print["of an expression with factor (l-h) into one with factor (-l+h)."]
Print["The expression is "]
                       L*(l-h)<=R*(l-h)
                       simplifying[%]
                       subst[lRestriction,%,"lRestriction"]
Print["Therefore, L <= R."]
Print["This completes the proof of Lemma 3.1. "]

"***********************************"
Print["Lemma 3.2: If dRestriction then R >= RLowerBound."]
"***********************************"

Print["Prove that R >= RLowerBound."]
Print["Since l > h, we prove instead that R*(l-h) >= RLowerBound*(l-h)."]
Print["The reason for this is to circumvent Matematica's transformation"]
Print["of an expression with factor (l-h) into one with factor (-l+h)."]
Print["The expression is "]
                       R*(l-h) >= RLowerBound*(l-h)
                       simplifying[%]
                       subst[dRestriction,%,"dRestriction"]
Print["Therefore, R >= RLowerBound."]
Print["This completes the proof of Lemma 3.2. "]
```

```
"************************************"
Print["Definitions"]
"************************************"

Print["The ColorLowerBound is "]
ColorLowerBound            = (h+2)*d -2*h
Print["NumColorsRX := Delta for R <= k is "]
NumColorsRX          = (d*(k+h+1)-(k+h))/(h+1)
Print["NumColorsLR := Delta for L <= k < R is "]
NumColorsLR          = (((2*d-4)*h+4*d-2)*l+2*(d-1)*k+(2-d)*h^2+(d-2)*h+4)/(2(l+1))

"************************************"
Print["Lemma 3.3: If dRestriction then Delta >= ColorLowerBound."]
"************************************"

Print["Prove that Delta >= ColorLowerBound. "]
Print["Case 1: R <= k."]
Print["We need to prove NumColorsRX >= ColorLowerBound. The expression is "]
                        NumColorsRX>=ColorLowerBound
                        simplifying[%]
Print["Since RLowerBound <= R and R <= k, then k >= RLowerBound."]
                        subst[k>=RLowerBound,%%,"k >= RLowerBound"]
Print["Therefore, Delta >= ColorLowerBound for Case 1"]
Print["Case 2: L <= k < R."]
Print["We need to prove NumColorsLR >= ColorLowerBound. The expression is "]
                        NumColorsLR >= ColorLowerBound
                        simplifying[%]
Print["By the conditions of the case L <= k."]
                        subst[k>=L,%%,"k >= L"]
Print["Therefore, Delta >= ColorLowerBound for Case 2"]
Print["This completes the proof of Lemma 3.3."]


"************************************"
Print["For Lemma 3.4 prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1 "]
"************************************"

Print["Case 1: R <= k."]
Print["We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1."]
Print["The expression is "]
           j+(h+1)(delta-(d-1))>=(j+k(d-1)+1)
           simplifying[%]
           subst[delta==NumColorsRX,%,"delta == NumColorsRX"]
Print["Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 1"]
```

```
Print["Case 2: L <= k < R."]
Print["We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1."]
Print["First we prove that"]
Print["j+(h+1)(delta-(d-1)) >= j+(k(h+1)(d-1))/(l+1)+R((d-1)(l-h))/(l+1)+1."]
Print["The expression is "]
       j+(h+1)(delta-(d-1))>=j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1
       subst[delta==NumColorsLR,%,"delta == NumColorsLR"]
Print["Therefore,"]
Print["j+(h+1)(delta-(d-1))>=j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1."]
Print["To complete the proof we show that"]
Print["j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1>=j+k(d-1)+1."]
Print["Since R > k, the proof reduces to showing that "]
                         j+(k*(h+1)*(d-1))/(l+1)+k*((d-1)*(l-h))/(l+1)+1>=j+k(d-1)+1
                         simplifying[%]
Print["Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 2."]
Print["This completes the proof of Lemma 3.4."]


"************************************"
Print["For Lemma 3.5 prove that"]
Print["d+2(2d-4)+((d-2)(h^2+h-6))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1."]
"************************************"

Print["Case 1: R <= k."]
Print["Prove that"]
Print["d+2(2d-4)+((d-2)(h^2+h-6))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1."]
Print["The expression is "]
       d+2*(2*d-4)+((d-2)*(h^2+h-6))/2+(delta-((h+2)d-2*h-1))*(l+1) >= k(d-1)+1
       simplifying[%]
       subst[delta==NumColorsRX,%,"delta == NumColorRX"]
       subst[k>=R,%,"k >= R"]
Print["Therefore, the expression holds for Case 1"]
Print["Case 2: L <= k < R."]
Print["Prove that"]
Print["d+2(2d-4)+((d-2)(h^2+h-6))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1."]
Print["The expression is "]
       d+2*(2*d-4)+((d-2)*(h^2+h-6))/2+(delta-((h+2)d-2*h-1))*(l+1) >= k(d-1)+1
       simplifying[%]
       subst[delta==NumColorsLR,%,"delta == NumColorLR"]
Print["Therefore, the expression holds for Case 2"]
Print["This completes the proof of Lemma 3.5"]

Quit
```

Output of "Mathematica" Program:

The symbol & in column 1 indicates continuation from previous “line”.

```
Mathematica 2.2 for HP9000 Series 700
Copyright 1988-94 Wolfram Research, Inc.
 -- Terminal graphics initialized --



        **************************************


            MACROS


        **************************************






        **************************************


        {1996, 7, 15, 17, 17, 6}


        **************************************

         Definitions

         **************************************

         The expression for R is
```

$$(1 + h)^2 + \frac{(1 + h)(-2 + 3h + h^2)}{2(-h + l)} + \frac{2 + 3h + h^3 - 2h^2 l}{2(-1 + d)(-h + l)}$$

The expression for L is

$$\frac{-(h + h^2)}{2(-1 + d)} + \frac{4 + h + h^2}{2} + \frac{l}{-1 + d}$$

dRestriction := The restriction for d is

$$d \geq \frac{-4 + 2h^2 + 2l}{-2 + 3h + h^2}$$

lRestriction := The restriction for l is

$$l \leq 2 - h^2 + \frac{d(-2 + 3h + h^2)}{2}$$

RLowerBound := A lower bound for R is

```
     2
    h       1 + h            2
-(------) + ------ + (1 + h)
```

```
  -1 + d    -1 + d
```

```
**************************************
```

Lemma 3.1: If dRestriction then L <= R.

```
**************************************
```

Prove that L <= R.

Since l > h, we prove instead that L*(l-h) >= R*(l-h).

The reason for this is to circumvent Matematica's transformation

of an expression with factor (l-h) into one with factor (-l+h).

The expression is

```
                          2              2
                  -(h + h )    4 + h + h       l
         (-h + l) (---------- + ---------- + ------) <=
                  2 (-1 + d)       2        -1 + d

                                                 2                3      2
                              2   (1 + h) (-2 + 3 h + h )   2 + 3 h + h  - 2 h  l
&    (-h + l) ((1 + h)  + ----------------------- + ---------------------)
                                    2 (-h + l)           2 (-1 + d) (-h + l)
```

SIMPLIFING THE EXPRESSION

```
                          2              2
                  -(h + h )    4 + h + h       l
         (-h + l) (---------- + ---------- + ------) <=
                  2 (-1 + d)       2        -1 + d

                                                 2                3      2
                              2   (1 + h) (-2 + 3 h + h )   2 + 3 h + h  - 2 h  l
&    (-h + l) ((1 + h)  + ----------------------- + ---------------------)
```

$$2\ (-h + l) \qquad\qquad 2\ (-1 + d)\ (-h + l)$$

------> USER MUST VERIFY THAT CONDITION [2 (-1 + d) > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

$$-4 + 2 d - 3 d h + 2 h^2 - d h^2 - 2 l + 2 d l - 3 d h l + 2 h^2 l - d h^2 l + 2 l^2 \le 0$$

SUBSTITUTING THE EXPRESSION ‘‘lRestriction’’

$$l \le 2 - h^2 + \frac{d\ (-2 + 3 h + h^2)}{2}$$

IN

$$-4 + 2 d - 3 d h + 2 h^2 - d h^2 - 2 l + 2 d l - 3 d h l + 2 h^2 l - d h^2 l + 2 l^2 \le 0$$

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 LessEqual 0] THAT EVALUATES TO True

Therefore, L <= R.

This completes the proof of Lemma 3.1.

****

Lemma 3.2: If dRestriction then R >= RLowerBound.

****

Prove that R >= RLowerBound.

Since l > h, we prove instead that R*(l-h) >= RLowerBound*(l-h).

The reason for this is to circumvent Matematica's transformation

of an expression with factor (l-h) into one with factor (-l+h).

The expression is

$$(-h + l) \left((1 + h)^2 + \frac{(1 + h) (-2 + 3 h + h^2)}{2 (-h + l)} + \frac{2 + 3 h + h^3 - 2 h^2 l}{2 (-1 + d) (-h + l)}\right) >= \left(-\left(\frac{h^2}{-1 + d}\right) + \frac{1 + h}{-1 + d} + (1 + h)^2\right) (-h + l)$$

SIMPLIFING THE EXPRESSION

$$(-h + l) \left((1 + h)^2 + \frac{(1 + h) (-2 + 3 h + h^2)}{2 (-h + l)} + \frac{2 + 3 h + h^3 - 2 h^2 l}{2 (-1 + d) (-h + l)}\right) >= \left(-\left(\frac{h^2}{-1 + d}\right) + \frac{1 + h}{-1 + d} + (1 + h)^2\right) (-h + l)$$

------> USER MUST VERIFY THAT CONDITION [2 (-1 + d) > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

$$4 - 2 d + 4 h + d h - 2 h^2 + 4 d h^2 - 2 h^3 + d h^3 - 2 l - 2 h l >= 0$$

SUBSTITUTING THE EXPRESSION ‘‘dRestriction’’

$$d >= \frac{-4 + 2 h^2 + 2 l}{-2 + 3 h + h^2}$$

IN

$$4 - 2 d + 4 h + d h - 2 h^2 + 4 d h^2 - 2 h^3 + d h^3 - 2 l - 2 h l >= 0$$

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, R >= RLowerBound.

This completes the proof of Lemma 3.2.

****

Definitions

****

The ColorLowerBound is

$-2 h + d (2 + h)$

NumColorsRX := Delta for R <= k is

```
-h - k + d (1 + h + k)
----------------------
        1 + h
```

NumColorsLR := Delta for L <= k < R is

```
                                 2
    4 + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k + (-2 + 4 d + (-4 + 2 d) h) l
&   ------------------------------------------------------------------------
                                     2 (1 + l)
```

****

Lemma 3.3: If dRestriction then Delta >= ColorLowerBound.

****

Prove that Delta >= ColorLowerBound.

Case 1: R <= k.

We need to prove NumColorsRX >= ColorLowerBound. The expression is

```
-h - k + d (1 + h + k)
---------------------- >= -2 h + d (2 + h)
        1 + h
```

SIMPLIFING THE EXPRESSION

```
-h - k + d (1 + h + k)
---------------------- >= -2 h + d (2 + h)
        1 + h
```

------> USER MUST VERIFY THAT CONDITION [1 + h > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

$$-d + h - 2 d h + 2 h^2 - d h^2 - k + d k \geq 0$$

Since RLowerBound <= R and R <= k, then k >= RLowerBound.

SUBSTITUTING THE EXPRESSION ‘‘k >= RLowerBound’’

$$k \geq -\left(\frac{h^2}{-1 + d}\right) + \frac{1 + h}{-1 + d} + (1 + h)^2$$

IN

$$-d + h - 2 d h + 2 h^2 - d h^2 - k + d k \geq 0$$

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, Delta >= ColorLowerBound for Case 1

Case 2: L <= k < R.

We need to prove NumColorsLR >= ColorLowerBound. The expression is

$$\begin{aligned}&(4 + (-2 + d) h + (2 - d) h^2 + 2 (-1 + d) k + \\ \& \quad &(-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l)) \geq -2 h + d (2 + h)\end{aligned}$$

SIMPLIFING THE EXPRESSION

$$\frac{4 + (-2 + d)\, h + (2 - d)\, h^2 + 2 (-1 + d)\, k + (-2 + 4 d + (-4 + 2 d)\, h)\, l}{2 (1 + l)}$$

$$\& \quad \geq -2 h + d (2 + h)$$

------> USER MUST VERIFY THAT CONDITION [2 (1 + l) > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

$$4 - 4 d + 2 h - d h + 2 h^2 - d h^2 - 2 k + 2 d k - 2 l \geq 0$$

By the conditions of the case L <= k.

SUBSTITUTING THE EXPRESSION ‘‘k >= L’’

$$k \geq \frac{-(h + h^2)}{2 (-1 + d)} + \frac{4 + h + h^2}{2} + \frac{l}{-1 + d}$$

IN

$$4 - 4 d + 2 h - d h + 2 h^2 - d h^2 - 2 k + 2 d k - 2 l \geq 0$$

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, Delta >= ColorLowerBound for Case 2

This completes the proof of Lemma 3.3.

```
**************************************

For Lemma 3.4 prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1

**************************************

Case 1: R <= k.

We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1.

The expression is

(1 - d + delta) (1 + h) + j >= 1 + j + (-1 + d) k

SIMPLIFING THE EXPRESSION

        (1 - d + delta) (1 + h) + j >= 1 + j + (-1 + d) k

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

         -d + delta + h - d h + delta h + k - d k >= 0

SUBSTITUTING THE EXPRESSION ``delta == NumColorsRX''

                 -h - k + d (1 + h + k)
        delta == ----------------------
                        1 + h

IN

        -d + delta + h - d h + delta h + k - d k >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS
```

```
THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

         Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 1


         Case 2: L <= k < R.

         We prove that j+(h+1)(delta-(d-1)) >= j+k(d-1)+1.

         First we prove that

         j+(h+1)(delta-(d-1)) >= j+(k(h+1)(d-1))/(l+1)+R((d-1)(l-h))/(l+1)+1.

         The expression is


         (1 - d + delta) (1 + h) + j >=

             (-1 + d) (1 + h) k
&    1 + j + ------------------ +
                   1 + l

                                                            2
                                   2   (1 + h) (-2 + 3 h + h )
&     ((-1 + d) (-h + l) ((1 + h)  + ----------------------- +
                                            2 (-h + l)

                     3      2
           2 + 3 h + h  - 2 h  l
&          ---------------------)) / (1 + l)
            2 (-1 + d) (-h + l)


SUBSTITUTING THE EXPRESSION ``delta == NumColorsLR''

                                                2
        delta == (4 + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k +

&       (-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l))

IN
```

```
        (1 - d + delta) (1 + h) + j >=

             (-1 + d) (1 + h) k
&    1 + j + ------------------ +
                   1 + l

                                                            2
                                     2   (1 + h) (-2 + 3 h + h )
&     ((-1 + d) (-h + l) ((1 + h)  + ---------------------- +
                                              2 (-h + l)

                      3      2
           2 + 3 h + h  - 2 h  l
&          ---------------------)) / (1 + l)
            2 (-1 + d) (-h + l)

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 2 (1 + l) > 0 ] HOLDS

THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

         Therefore,

         j+(h+1)(delta-(d-1))>=j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1.

         To complete the proof we show that

         j+(k*(h+1)*(d-1))/(l+1)+R*((d-1)*(l-h))/(l+1)+1>=j+k(d-1)+1.

         Since R > k, the proof reduces to showing that


                 (-1 + d) (1 + h) k   (-1 + d) k (-h + l)
         1 + j + ------------------ + ------------------- >=
                       1 + l                 1 + l

&    1 + j + (-1 + d) k


SIMPLIFING THE EXPRESSION

                (-1 + d) (1 + h) k   (-1 + d) k (-h + l)
```

```
        1 + j + ------------------ + ------------------- >= 1 + j + (-1 + d) k
                      1 + l                 1 + l
```

------> USER MUST VERIFY THAT CONDITION [1 > 0] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS

[0 GreaterEqual 0] THAT EVALUATES TO True

Therefore, j+(h+1)(delta-(d-1))>=(j+k(d-1)+1) for Case 2.

This completes the proof of Lemma 3.4.

****

For Lemma 3.5 prove that

d+2(2d-4)+((d-2)(h^2+h-6))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1.

****

Case 1: R <= k.

Prove that

d+2(2d-4)+((d-2)(h^2+h-6))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1.

The expression is

```
                                                    2
                           (-2 + d) (-6 + h + h )
          d + 2 (-4 + 2 d) + ---------------------- +
                                     2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k
```

SIMPLIFING THE EXPRESSION

```
                                                  2
                         (-2 + d) (-6 + h + h )
        d + 2 (-4 + 2 d) + ---------------------- +
                                    2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k

------> USER MUST VERIFY THAT CONDITION [ 2 > 0 ] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS


                                             2      2
          -4 + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l -

&     4 d l + 2 delta l + 4 h l - 2 d h l >= 0


SUBSTITUTING THE EXPRESSION ``delta == NumColorRX''

                 -h - k + d (1 + h + k)
        delta == ----------------------
                          1 + h

IN

                                           2      2
        -4 + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l - 4 d l +

&     2 delta l + 4 h l - 2 d h l >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS

THE RESULTING EXPRESSION IS


                                          3      3
          (-4 + 2 d - 4 h + d h - 2 h  + d h  + 2 h k - 2 d h k + 2 l -

                                          2          2
&       2 d l + 4 h l - 4 d h l + 4 h  l - 2 d h  l - 2 k l + 2 d k l) /

&     (1 + h) >= 0
```

```
SUBSTITUTING THE EXPRESSION ‘‘k >= R’’

                                                 2                3      2
                    2   (1 + h) (-2 + 3 h + h )   2 + 3 h + h  - 2 h  l
        k >= (1 + h)  + ----------------------- + ---------------------
                              2 (-h + l)           2 (-1 + d) (-h + l)

IN

                                 3      3
        (-4 + 2 d - 4 h + d h - 2 h  + d h  + 2 h k - 2 d h k + 2 l - 2 d l +

                                 2          2
&       4 h l - 4 d h l + 4 h  l - 2 d h  l - 2 k l + 2 d k l) / (1 + h) >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 + h > 0 ] HOLDS

THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

          Therefore, the expression holds for Case 1

          Case 2: L <= k < R.

          Prove that

          d+2(2d-4)+((d-2)(h^2+h-6))/2+(delta-((h+2)d-2h-1))(l+1) >= k(d-1)+1.

          The expression is


                                                     2
                              (-2 + d) (-6 + h + h )
          d + 2 (-4 + 2 d) + ---------------------- +
                                        2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k


SIMPLIFING THE EXPRESSION
```

```
                                               2
                     (-2 + d) (-6 + h + h )
d + 2 (-4 + 2 d) + ---------------------- +
                               2

&     (1 + delta + 2 h - d (2 + h)) (1 + l) >= 1 + (-1 + d) k

------> USER MUST VERIFY THAT CONDITION [ 2 > 0 ] HOLDS

THE RESULTING EXPRESSION WITH ALL TERMS ON THE LEFT SIDE IS


                                          2       2
          -4 + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l -

&     4 d l + 2 delta l + 4 h l - 2 d h l >= 0


SUBSTITUTING THE EXPRESSION ``delta == NumColorLR''

                                             2
        delta == (4 + (-2 + d) h + (2 - d) h  + 2 (-1 + d) k +

&       (-2 + 4 d + (-4 + 2 d) h) l) / (2 (1 + l))

IN

                                        2       2
        -4 + 2 delta + 2 h - d h - 2 h  + d h  + 2 k - 2 d k + 2 l - 4 d l +

&     2 delta l + 4 h l - 2 d h l >= 0

------> VALID SUBSTITUTION

------> USER MUST VERIFY THAT CONDITION [ 1 > 0 ] HOLDS

THE RESULTING EXPRESSION IS

 [ 0 GreaterEqual 0 ] THAT EVALUATES TO True

          Therefore, the expression holds for Case 2

          This completes the proof of Lemma 3.5
```

References

[1] Gonzalez, T. F., "Improved Approximation Algorithms for Multimessage Multicasting," UCSB Department of Computer Science, Technical Report TRCS-96-16, July 1996.