Isoperimetric Number of the Cartesian Product of Graphs and Paths

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Abstract

We prove that the isoperimetric number of $P_k \times G_k$, the Cartesian product of the path P_k and a connected graph with k vertices, is equal to the isoperimetric number of P_k itself. At the same time we construct an infinite family of graphs that shows that this is not true for $P_k \times G$ where G has more than k vertices, even if G is a tree.

Keywords: Isoperimetric number, bisection width, path, array, Cartesian product graph.

1 Introduction

Given a graph G and a subset X of its vertices, let ∂X denote the *edge-boundary* of X: i.e. the set of edges which connect vertices in X with vertices not in X. The *isoperimetric number* of G is defined as

$$i(G) = \min_{1 \le |X| \le \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|}.$$

As examples, $i(K_k) = \lceil \frac{k}{2} \rceil$ for the complete graph K_k, $i(C_k) = 2/\lfloor \frac{k}{2} \rfloor$ for the k-cycle C_k, and $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ for the path (chain) P_k on k vertices. We refer the reader to Mohar [9], for a discussion of basic results and various interesting properties of $i(G)$. Works by Bezrukov [2], Bollobás and Leader [3, 4], Ahlswede and Bezrukov [1], Riordan [11], also contain recent results on isoperimetric properties of various classes of graphs.

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A related quantity to $i(G)$ is the *bisection width*. The bisection width $bw(G)$ of a graph G is the minimum number of edges which must be removed from G in order to split it into two parts with equal (within one, when the number of vertices of G is odd) number of vertices. The isoperimetric number of a graph establishes a lower bound for its bisection width.

The *Cartesian product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent if and only if u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. Product graphs are important since many interesting graphs are products of simpler graphs, and sometimes methods of analysis can be lifted from the constituent graphs to their products [2, 5]. Among families of graphs that are products are the *d-dimensional hypercube* Q_d, which is the d-fold product of K_2, *d-dimensional k-torus* T_k^d, which is the d-fold product of C_k, and the *d-dimensional k-array* A_k^d, which is the d-fold product of P_k. In general

$$i(G \times H) \leq \min\{i(G), i(H)\} \tag{1}$$

(see [9]), and thus product graphs do not always behave nicely with respect to isoperimetric numbers of their factors. There are exceptions however: Mohar [9] showed that $i(K_{2n} \times G) = \min\{n, i(G)\}$ whenever G has an even number of vertices.

Our basic result is that $i(P_k \times G) = i(P_k)$ for a connected graph G on k vertices, whereas equality fails if G has more than k vertices, even if G is a tree.

1.1 Multidimensional arrays

Edge-isoperimetric properties of multidimensional arrays and its varieties have been studied by many authors. This problem is related to the *maximum induced edge problem* where, for a given m, a subset of vertices with the largest number of induced edges is sought among all m-element subsets [4]. The two problems are equivalent for regular graphs, but not for multidimensional arrays.

The maximum induced edge problem under *Hamming metric* (hence the isoperimetric number problem, because of the regularity of the Hamming metric) was solved by Harper [6] on the discrete cube and extended by Lindsey [8] to $P_{k_1} \times \cdots \times P_{k_d}$. In both instances, there is a nested structure of solutions, and the first m vertices in *lexicographical order* constitute a solution. The analogue for the even discrete torus appears in Riordan [11]. The maximum induced edge problem for multidimensional arrays was solved by Bollobás and Leader [4]. This work also contains bounds for the isoperimetric number problem. Ahlswede and Bezrukov [1] solved the isoperimetric number problem for $P_\infty \times \cdots \times P_\infty$ where the minimum is taken over all non-empty finite subsets, and gave a solution for $P_{k_1} \times P_{k_2}$ for arbitrary k_1, k_2 as well.

1.2 Motivation

Our initial motivation in this work was to give an alternate proof of the lower bound

$$bw(A_k^d) \geq \frac{k^d - 1}{k - 1} \tag{2}$$

for odd k. This was proved by Nakano [10] by an embedding of a d-dimensional k-clique into A_k^d. Prior to this Leighton [7] showed that $bw(A_k^d) \geq k^{d-1}$ when k is even. The proof involves embedding of a complete graph into A_k^d. However, this embedding does not give a tight bound when k is odd. One could attack the problem by first showing that $i(A_k^d) = 2/(k-1)$ for odd k, then the bisection width bound would follow from

$$\frac{bw(A_k^d)}{\frac{k^d-1}{2}} \geq \frac{2}{k-1} \;\Rightarrow\; bw(A_k^d) \geq \frac{k^d - 1}{k - 1}. \tag{3}$$

Mohar [9] showed that $i(P_k \times P_n) = \min\{i(P_k), i(P_n)\}$, and therefore $i(A_k^2) = 2/(k-1)$. Since $A_k^d = P_k \times A_k^{d-1}$, the computation of $i(A_k^d)$ naturally leads to the study of isoperimetric numbers of product graphs of the form $i(P_k \times G)$ where G is an arbitrary graph (in the most general case), and $i(P_k \times T)$ where T is a tree (in a weaker case). General results on graph products based on the second smallest eigenvalue of the Laplacian [9], or the bound

$$\frac{1}{2}m \leq i(G_1 \times G_2 \times \cdots \times G_m) \leq m$$

where $m = \min\{i(G_1), i(G_2), \cdots, i(G_m)\}$ reported by Chung and Tetali [5] do not give the tight enough lower bound for $i(A_k^d)$.

The outline of this paper is as follows. In section 2, we prove $i(P_k \times G_k) = i(P_k)$ where G_k is any connected graph with k vertices. In section 3, we consider the isoperimetric number of the product graph $P_k \times G$ where G is an arbitrary connected graph and show that equality does not carry over to general graphs. First we construct a simple counterexample and then extend it to an infinite family of graphs. Section 4 concludes with remarks.

2 The Product Graph $P_k \times G_k$

Let us first consider the Cartesian product of the path P_k with G_k, where G_k is a connected graph on k vertices.

Theorem 1 *$i(P_k \times G_k) = 1/\lfloor k/2 \rfloor$ for any connected graph G_k on k vertices.*

Proof We prove the theorem for odd k, i.e. $i(P_k \times G_k) = 2/(k-1)$, as this is the interesting case. First note that among all connected graphs with k vertices, the isoperimetric number of P_k is the smallest. Thus by (1)

$$i(P_k \times G_k) \leq \min\{i(P_k), i(G_k)\} = i(P_k) = \frac{2}{k-1},$$

and to prove the theorem we only need to show $i(P_k \times G_k) \geq 2/(k-1)$. Let $V(P_k) = \{1, 2, \ldots, k\}$ and $X \subseteq V(P_k \times G_k)$ with $1 \leq |X| \leq (k^2-1)/2$. For $i = 1, 2, \ldots, k$ let $X_i = X \cap (V(G_k) \times \{i\})$. Thus X is the disjoint union of $X_1, X_2, \ldots, X_k$. We partition the set of edges in the boundary as $\partial X = \partial_P X \cup \partial_G X$ where $\partial_P X$ is the set of *interlevel* boundary edges, i.e. edges lying in copies of P_k in the product graph, and $\partial_G X$ is the set of *intralevel* boundary edges, i.e. edges internal to each copy of G_k. This is illustrated in Figure 1. Define N_0 and N_k by $N_0 = |\{X_i \mid |X_i| = 0, 1 \leq i \leq k\}|$ and

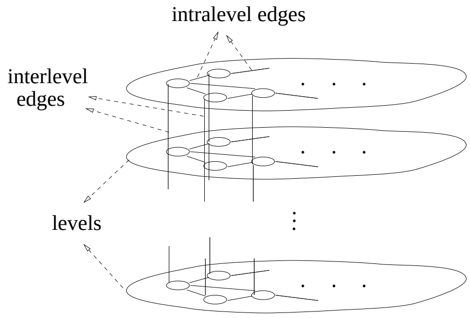

Figure 1: The Cartesian product $P_k \times G$.

$N_k = |\{X_i \mid |X_i| = k, 1 \leq i \leq k\}|$. Consider the intralevel edges $\partial_G X_i$ in the boundary of X_i. If $|X_i| = 0$ or $|X_i| = k$ then $|\partial_G X_i| = 0$, otherwise $|\partial_G X_i| \geq 1$. Similarly, the contribution of the interlevel edges between X_i and X_{i+1} to ∂X is the symmetric difference of these two sets $X_i \,\Delta\, X_{i+1}$. Thus

$$|\partial X| = |\partial_G X| + |\partial_P X| \geq k - N_0 - N_k + \sum_{i=1}^{k-1} |X_i \,\Delta\, X_{i+1}|.$$

By the triangle inequality, the sum of symmetric differences is minimum when $|X_i|$'s are in sorted (increasing or decreasing) order. Thus $|\partial X| \geq k - N_0 - N_k + |X_1| - |X_k|$, and to prove the theorem it suffices to prove the inequality

$$k - N_0 - N_k + |X_1| - |X_k| \geq \frac{2}{k-1}|X|, \tag{4}$$

subject to

1. $k \geq |X_1| \geq |X_2| \geq \cdots \geq |X_k| \geq 0,$
2. $|X| = |X_1| + |X_2| + \cdots + |X_k|,$
3. $1 \leq |X| \leq (k^2 - 1)/2.$

Proof of (4) is broken down into 4 cases according to possible values of N_0 and N_k.

Case (1) $N_0 = 0,\ N_k = 0$: In this case, (4) reduces to

$$k + |X_1| - |X_k| \geq \frac{2}{k-1}|X|.$$

First suppose that not all $|X_i|$ are equal. Then the inequality holds since

$$k + |X_1| - |X_k| \geq k + 1 = \frac{2}{k-1}\frac{k^2-1}{2} \geq \frac{2}{k-1}|X|.$$

If all $|X_i|$ are equal then $|X| \leq k\,(k-1)/2$ and

$$k + |X_1| - |X_k| = k = \frac{2}{k-1}k\frac{k-1}{2} \geq \frac{2}{k-1}|X|.$$

Case (2) $N_0 > 0,\ N_k = 0$: In this case the first condition becomes

$$k > |X_1| \geq |X_2| \geq \cdots \geq |X_l| > 0 = |X_{l+1}| = \cdots = |X_k|$$

And (4) becomes

$$l + |X_1| \geq \frac{2}{k-1}(|X_1| + |X_2| + \cdots + |X_l|)$$

Thus, it is sufficient to prove

$$l + |X_1| \geq \frac{2}{k-1}\,l\,|X_1| \tag{5}$$

or equivalently, $(k-1)l + (k-1)|X_1| \geq 2\,l\,|X_1|$. Since $l \leq k-1$ and $|X_1| \leq k-1$, we have

$$(k-1)l + (k-1)|X_1| \geq l^2 + |X_1|^2.$$

But $l^2 + |X_1|^2 \geq 2\,l\,|X_1|$ since $(l - |X_1|)^2 \geq 0$, and (5) follows.

Case (3) $N_0 = 0,\ N_k > 0$: Now the $|X_i|$ satisfy

$$k = |X_1| \geq |X_2| \geq \cdots \geq |X_k| > 0,$$

while the inequality we want to prove becomes

$$k - N_k + k - |X_k| \geq \frac{2}{k-1}|X|.$$

It suffices to prove

$$2k - N_k - |X_k| \geq k + 1 \tag{6}$$

since $|X| \leq (k^2 - 1)/2$. This condition on $|X|$ also forces $N_k \leq (k-1)/2$ and $|X_k| \leq (k-1)/2$, and (6) follows.

Case (4) $N_0 > 0,\ N_k > 0$: As in the previous case, it is sufficient to prove

$$k - N_0 - N_k + k \geq k + 1$$

which obviously holds for $N_0 + N_k \leq k - 1$. For $N_0 + N_k = k$, $|X| \leq k\,(k-1)/2$. Thus, we have

$$2k - N_0 - N_k = k = \frac{2}{k-1}\ k\ \frac{k-1}{2} \geq \frac{2}{k-1}|X|.$$

Therefore inequality (4) holds in all cases, and the theorem follows. □

At this point, consider $i(P_k \times G_n)$ for a connected graph G_n with arbitrary number of vertices n. It is tempting to conjecture that Theorem 1 extends to this general case as well, i.e. $i(P_k \times G) = 2/(k-1)$. Of course, in view of (1), this can only hold for G with $i(G) \geq 2/(k-1)$. We show in the next section that even for such graphs the equality does not hold.

3 The Product Graph $P_k \times G$

We start with an example for $k = 5$. Consider the graph $G = G_{11}$ on 11 vertices shown in Figure 2. By inspection, an isoperimetric set for G

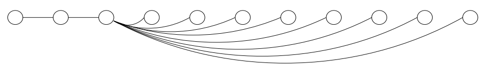

Figure 2: The graph $G = G_{11}$.

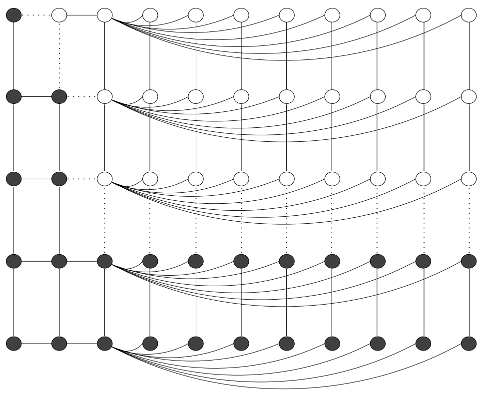

Figure 3: $P_5 \times G_{11}$, subset X and the boundary edges ∂X.

consists of the two leftmost vertices in Figure 2, and therefore $i(G) = 1/2$. If $k = 5$ then $i(G) = 1/2 \geq i(P_5) = 2/(5-1) = 1/2$, and $i(G)$ satisfies the necessary condition mentioned above. The product graph $P_5 \times G_{11}$ is shown in Figure 3. Assume X is the subset indicated by the dark vertices. Then $|X| = 27 \leq (5 \times 11 - 1)/2$ as required. The dotted edges comprise the boundary ∂X and $|\partial X| = 13$. Thus

$$i(P_5 \times G_{11}) \leq |\partial X|/|X| = 13/27 < 2/(k-1) = 1/2.$$

The following proposition provides an infinite family of graphs, generalizing this counterexample.

Proposition 1 *For any odd number k, there exists an infinite family of graphs G_g with $i(G_g) \geq 2/(k-1)$ and $i(P_k \times G_g) < 2/(k-1)$.*

Proof Suppose $k = 2m+1$. Consider the graph G_g on $g = m + m' + 1$ vertices for $m' \geq m$ obtained by joining the path P_m and the star graph $K_{1,m'}$ as shown in Figure 4. We pick m' so that g is odd. Since $m' \geq m$, an isoperimetric set for G_g is the first m vertices on the left in Figure 4. Thus $i(G_g) = 1/m$. The graph $P_k \times G_g$ is shown in Figure 5. It has $(2m+1)g$

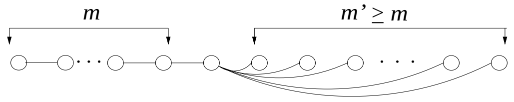

Figure 4: The base graph for the general case.

vertices. Consider the subset X represented by dark vertices in Figure 5. X

is defined by taking $X_1, X_2, \cdots, X_m$ to be G, $X_{m+1}, X_{m+2}, \cdots, X_{2m}$ to be the vertices on P_m in the corresponding copy of G in $P_k \times G_g$, and X_{2m+1} to be the singleton as indicated in Figure 5. The dotted edges are the edges in

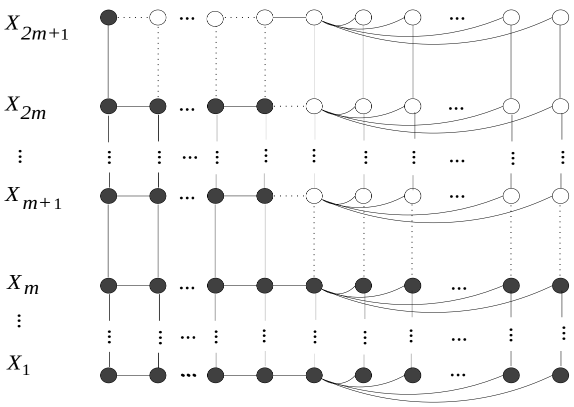

Figure 5: The structure of the product graph for the general case.

the boundary ∂X. Then $|X| = mg + m^2 + 1$ and $|\partial X| = g - 1 + m + 1 = m + g$. Furthermore whenever m' is chosen so that $g \geq 2m^2 + 3$, the inequality

$$|X| = mg + m^2 + 1 \leq \frac{(2m+1)g - 1}{2}$$

holds. Therefore

$$\frac{|\partial X|}{|X|} = \frac{m+g}{mg + m^2 + 1} < \frac{1}{m} = \frac{2}{(k-1)},$$

and $i(P_k \times G_g) < 2/(k-1)$. □

Note that the graphs G_g are trees. Hence even for trees T with $i(T) = 2/(k-1)$, it is possible to have $i(P_k \times T) \neq i(P_k)$, unless T has k vertices, as guaranteed by Theorem 1.

4 Conclusion and Remarks

We considered the isoperimetric number of graphs of the form $P_k \times G$. If G is a connected graph on k nodes, then $i(P_k \times G) = i(P_k)$, whereas equality fails in general if $i(G) = i(P_k)$ but G has more than k vertices. For every odd k, we constructed an infinite family of graphs (actually trees) G_g for which $i(P_k \times G_g) < i(P_k)$.

Our motivation for studying product graphs with paths is the bound (2) on the bisection width of A_k^d, and the inequality (3) in terms of its isoperimetric number. Our result shows that for odd k, the calculation of $i(A_k^d) = i(P_k \times A_k^{d-1})$ does not follow from a general result on isoperimetric numbers of Cartesian product graphs $P_k \times G$, and the fact that $G = A_k^{d-1}$ is necessary.

Acknowledgement

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