The Isoperimetric Number of d-dimensional k-ary Arrays

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Abstract

The d–dimensional k–ary array A_k^d is the k–fold Cartesian product graph of the path P_k on k vertices. We show that the (edge) isoperimetric number of A_k^d is given by $i(A_k^d) = i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ for all k and d, and identify the cardinalities and the structure of the isoperimetric sets. For k odd, the cardinalities of the isoperimetric sets are $\frac{1}{2}(k^d-1), \frac{1}{2}(k^d-k), \ldots, \frac{1}{2}(k^d-k^{d-1})$, whereas every isoperimetric set in A_k^d has cardinality $\frac{1}{2}k^d$ when k is even.

1 Introduction

Given a graph G and a subset X of its vertices, let ∂X denote the *edge–boundary* of X; the set of edges which connect vertices in X with vertices in $V(G) \setminus X$. The *isoperimetric number* of G is defined as

$$i(G) = \min_{1 \le |X| \le \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|} \ . \tag{1}$$

That is, the set of vertices of G is partitioned into two nonempty sets and that the ratio of number of edges between the two parts and number of vertices in the smaller one is minimized over all such partitions. As examples of isoperimetric numbers, $i(K_k) = \lceil \frac{k}{2} \rceil$ for the complete graph K_k, $i(C_k) = 2/\lfloor \frac{k}{2} \rfloor$ for the k–vertex cycle C_k, and $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ for the k–vertex path (chain) P_k. A subset X which achieves the minimum ratio in (1) is called an *isoperimetric set*. We refer the reader to Mohar [11] or Chung [6] for a discussion of basic results and various interesting properties of $i(G)$.

The *d–dimensional k–ary array* A_k^d is an undirected graph with k^d nodes labeled by the integers from 0 to k^d-1. Two nodes in A_k^d are connected by an edge if and only if the k–ary representations of their labels differ in exactly one digit and the absolute value of the difference in that digit is 1. Figure 1 illustrates a 2–dimensional 4–ary array A_4^2. The Cartesian product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent if and only if u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. A *multidimensional array* is a Cartesian product of paths of varying length, i.e. $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$. Thus A_k^d is a special type of multidimensional array in which $k_1 = k_2 = \cdots = k_d = k$.

In this paper, we investigate isoperimetric properties of d–dimensional k–ary arrays for arbitrary d and $k \ge 2$. Specifically, we prove the following.

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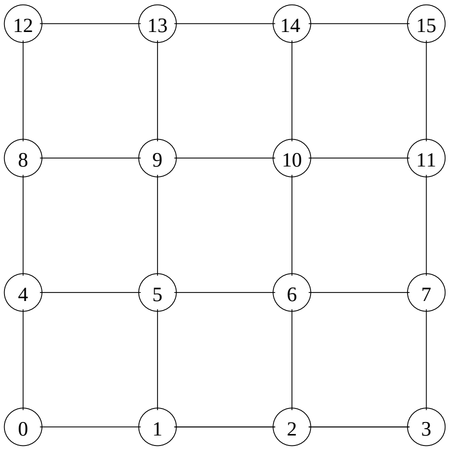

Figure 1: The 2–dimensional 4–ary array A_4^2.

Theorem 1 *The isoperimetric number of the d–dimensional k–ary array* A_k^d *is given by*

$$i(A_k^d) = i(P_k) = \begin{cases} 2/k & k \text{ } even, \\ 2/(k-1) & k \text{ } odd. \end{cases} \tag{2}$$

As a byproduct of the proof, we also show that there are exactly d distinct cardinalities

$$\tfrac{1}{2}(k^d - 1),\ \tfrac{1}{2}(k^d - k),\ \ldots\ ,\ \tfrac{1}{2}(k^d - k^{d-1})$$

of isoperimetric sets in the d–dimensional k–ary array A_k^d when k is odd. In contrast, every isoperimetric set has cardinality $k^d/2$ when k is even. It is worth noting that as in many families of graphs studied in the literature, the k *odd* case is substantially more difficult to prove than the k *even* case.

1.1 Motivation

The notion of isoperimetric number of a graph G serves as a measure of connectivity of G as it quantifies the minimal interaction between a set of vertices X and its complement $\overline{X}$ in terms of the number of edges between them. This idea is also important in algorithm design. For instance, the notion of isoperimetric number is implicit in the divide-and-conquer strategy in graph algorithms. To illustrate, consider an algorithm which adopts divide-and-conquer strategy where the set of vertices of the underlying graph is split into two "fairly balanced" parts such that the algorithm can be run on the two corresponding subgraphs recursively, and the results are combined to get the solution for the original problem. The combining of results at the last step needs to be carried out with minimal effort if such a scheme is expected to be efficient. The trick is to split the graph in such a way as to keep the interaction between the two partitions (in terms of the number of edges in the boundary) as small as possible. The isoperimetric number is closely related to the notion of *bisection width* $bw(G)$ of a graph G, which is the minimum number of edges that must be removed from the graph in order to split $V(G)$ into two *equal–sized* (within one) subsets. The isoperimetric number of a graph establishes a lower bound for its bisection width. For instance, one can give an alternate proof of the known lower bound

$$bw(A_k^d) \geq \begin{cases} k^{d-1} & k \ even, \\ \frac{k^d-1}{k-1} & k \ odd \end{cases} \tag{3}$$

using the formula (2) for the isoperimetric number, since as a special case

$$\frac{bw(A_k^d)}{\lfloor \frac{k^d}{2} \rfloor} \geq i(A_k^d).$$

For k even, (3) was proved by Leighton [9] by an embedding method. Nakano [12] also used an embedding technique to prove the odd case in inequality (3). In this paper, we extend these two techniques to get tight edge–isoperimetric lower bounds which lead to the exact formula (2) for $i(A_k^d)$ for all k and d.

1.2 Outline

The outline of the rest of this paper is as follows. In section 2 we summarize previous work on isoperimetric properties of various families of product graphs. The proof of our main result appears in section 3. We treat the cases of even and odd k separately, and in each of these cases, we give tight upper and lower bounds that prove the formula (2) for $i(A_k^d)$. In section 4 we give the possible cardinalities of the isoperimetric sets in A_k^d, describe their recursive construction, and conclude the paper.

2 A Summary of Previous Work

There has been a significant amount of research in the area of isoperimetric bounds on various popular classes of graphs such as arrays and tori. The notion of isoperimetric number of a graph is related to the theory of extremal sets in graphs. An *extremal set* of a graph for a given m is, in a broad sense, a configuration of m vertices with

i) minimum number of boundary edges, or

ii) maximum number of spanned edges

among all m–vertex subsets. The problem of finding extremal sets of the first (or, second) type is called *the minimum–boundary–edge problem* (or, *the maximum–induced–edge problem*). It can be shown that the minimum–boundary–edge and the maximum–induced–edge problems are equivalent for regular graphs [5]. Let ∂_m denote the number of boundary edges of an extremal set of the first type for any given m in a graph. If we can compute ∂_m for any given m in a graph G then we can easily obtain $i(G)$ by determining the minimum ratio ∂_m/m for each m in $1 \leq m \leq |V(G)|/2$.

The maximum–induced–edge problem for the hypercube (hence the minimum–boundary–edge problem, because of its regularity) was solved by Harper [8] and extended by Lindsey [10] to the d–dimensional k–ary clique which we shall define formally in the next section. In both instances, there is a nested structure of solutions, and the first m vertices in *lexicographical order* constitute an extremal set. The analogue for the even discrete torus appears in Riordan [13]. The maximum–induced–edge problem for the d–dimensional k–ary array A_k^d was first solved by

Bollobás and Leader [5]. Since A_k^d is not regular, this is not helpful for the minimum–boundary–edge problem. Their result was later extended to multidimensional arrays by Ahlswede and Bezrukov [1] who also gave a solution for $P_{k_1} \times P_{k_2}$ for the minimum–boundary–edge problem. The first nontrivial bounds on the minimum–boundary–edge problem for the d–dimensional k–ary arrays were given by Bollobás and Leader [5]. The bounds obtained are not sharp enough to get $i(A_k^d)$ exactly however. Wang and Wang [14] solved the minimum–boundary–edge problem problem for $P_\infty \times \cdots \times P_\infty$, i.e. the *$d$–dimensional infinite grid*, where the minimum is taken over all nonempty finite subsets. They also gave a simple ordering of the vertices in which the first m vertices constitute an extremal set minimizing the edge–boundary.

A natural approach for the exact calculation of $i(A_k^d)$ is to try to exploit the recursive Cartesian product structure $A_k^d = P_k \times A_k^{d-1}$. For instance, Mohar [11] gave a proof for $i(P_{k_1} \times P_{k_2}) = \min\{i(P_{k_1}), i(P_{k_2})\}$. It can be proved that in general $i(P_k \times G) = i(P_k)$ where G is any connected graph with k vertices [2]. These results indicate that (2) holds for $d = 2$. However, to extend them to get the desired bound for $i(P_k \times A_k^{d-1})$ directly does not seem possible. It was shown in [2], for instance, that $i(P_k \times G) \neq i(P_k)$ if G has more than k vertices even when $i(G) \geq i(P_k)$ (it is already known that $i(P_k \times G) < i(P_k)$ if $i(G) < i(P_k)$ as explained in the next section). General results on graph products based on the second smallest eigenvalue of the Laplacian [11], or the bound $\frac{1}{2}m \leq i(G_1 \times G_2 \times \cdots \times G_n) \leq m$ where $m = \min\{i(G_1), i(G_2), \cdots, i(G_n)\}$ reported by Chung and Tetali [7] do not yield the desired sharp lower bound for $i(A_k^d)$, either.

3 The Isoperimetric Number of A_k^d

In this section we prove the two cases in Theorem 1 by showing that each expression on the right–hand side of equation (2) is an upper and a lower bound for $i(A_k^d)$. The upper bound argument does not use the parity of k, and so there are no cases. The lower bound calculation is split into separate cases, even k and odd k.

3.1 The Upper Bound

To prove the upper bound, we utilize a general inequality given in [11] on the isoperimetric number of a Cartesian product graph, i.e.

$$i(G \times H) \leq \min\{i(G), i(H)\}. \tag{4}$$

For completeness, we sketch the proof. Consider $Y \subseteq V(G \times H)$ where

$$Y = \{(u, v) \mid u \in X,\ X \subseteq V(G) \text{ and}$$

Then we have

$$i(G \times H) \leq \frac{|\partial Y|}{|Y|} = \frac{|V(H)|\ |\partial X|}{|V(H)|\ |X|} = i(G),$$

and similarly $i(G \times H) \leq i(H)$. Thus we have the following corollary.

Corollary 1 *For the d–dimensional k–ary array A_k^d, we have $i(A_k^d) \leq 1/\lfloor \frac{k}{2} \rfloor$.*

Proof By taking $G = P_k$ and $H = A_k^{d-1}$ in (4), we have $i(A_k^d) \leq \min\{i(P_k), i(A_k^{d-1})\}$ which implies $i(A_k^d) \leq 1/\lfloor \frac{k}{2} \rfloor$ since $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$. □

3.2 The Lower Bound for Even Case

We first state our claim formally in the following proposition.

Proposition 1 *For the d–dimensional k–ary array A_k^d with k even, $i(A_k^d) \geq 2/k$.*

Proof We prove the lower bound for even k by extending the embedding technique of Leighton [9] who used it to obtain a lower bound for $bw(A_k^d)$ for even k. Given a d–dimensional k–ary array A_k^d where k is even, we embed into A_k^d the k^d–node directed complete graph K_{k^d} with the vertex set $\{0, \ldots, k^d - 1\}$. Any two distinct vertices u and $v \in V(K_{k^d})$ are connected by the directed edges (u, v) and (v, u).

The edge from node $u = (u_1, u_2, \cdots, u_d)$ to node $v = (v_1, v_2, \cdots, v_d)$ of the complete graph is routed through the path

$$(u_1, u_2, \cdots, u_d) \rightarrow (v_1, u_2, \cdots, u_d) \rightarrow (v_1, v_2, u_3 \cdots, u_d) \rightarrow \cdots \rightarrow (v_1, v_2, \cdots, v_d)$$

in the array. That is, when routing the edge from u to v, following edges in A_k^d, we first "correct" the value of u along the dimension with the smallest index that is different from the value in v at that index until the two values become equal. Then, we correct the value of the next smallest index where they differ and so on. The process stops when all the dimensions have eventually been corrected.

When k is even, at most $k^{d+1}/2$ edges of the complete graph are routed through a given edge in the array (see [9], page 225). In other words, removal of an edge in the array is equivalent to disconnecting at most $k^{d+1}/2$ edges in the complete graph.

Now we prove that $|\partial X|/|X| \geq 2/k$ for any $X \subseteq V(A_k^d)$ where $1 \leq |X| \leq k^d/2$. Consider such a subset X. Also, let X' be the corresponding subset of vertices in $V(K_{k^d})$ under the embedding. Then X' has boundary $\partial X'$ with $2|X'|(k^d - |X'|)$ edges in the complete graph since each vertex in X' is connected with every vertex in the remaining $k^d - |X'|$ vertices by two (directed) edges. That is, $|\partial X'| = 2|X'|(k^d - |X'|)$. We then have

$$|\partial X| \geq \frac{|\partial X'|}{\frac{k^{d+1}}{2}} \quad \Rightarrow \quad \frac{|\partial X|}{|X|} \geq \frac{|\partial X'|}{\frac{k^{d+1}}{2}|X|} \geq \frac{2|X'|(k^d - |X'|)}{\frac{k^{d+1}}{2}|X|}$$

But

$$\frac{2|X'|(k^d - |X'|)}{\frac{k^{d+1}}{2}|X|} \geq \frac{2}{k} \tag{5}$$

since this is an immediate consequence of the fact that $|X| = |X'|$ and $|X| \leq k^d/2$. □

3.3 The Lower Bound for Odd Case

In this section, we prove the following proposition.

Proposition 2 *For the d–dimensional k–ary array A_k^d with $k > 1$ odd, $i(A_k^d) \geq 2/(k-1)$.*

Unfortunately, the technique used for even k does not yield a tight lower bound when k is odd. Specifically, with the same embedding method, one can at best get

$$i(A_k^d) \geq \frac{2(k^d + 1)}{k^{d+1} - k^{d-1}}$$

for odd k, and the right hand side is a smaller lower bound than the desired $2/(k-1)$. Instead, we prove this case by extending the embedding technique of Nakano [12]. Before going into the proof, we shall first give a characterization of graphs based on *linear layouts* and develop some notation to facilitate our treatment. Our notation and terminology are similar to those used in [12].

3.4 Linear Layouts

A *linear layout* L of a graph G is a one-to-one mapping between the vertices in the graph and the numbers $\{0, \ldots, |V(G)|-1\}$, i.e., $L : V(G) \rightarrow \{0, 1, \ldots, |V(G)|-1\}$ is a bijection. One can think of this mapping as assigning each node a position on the number line between 0 and $|V(G)|-1$ as shown in Figure 2. For a d–dimensional k–ary array A_k^d, the identity mapping I which assigns each vertex to the value of its own label is called the *label–order layout*. We refer to the region

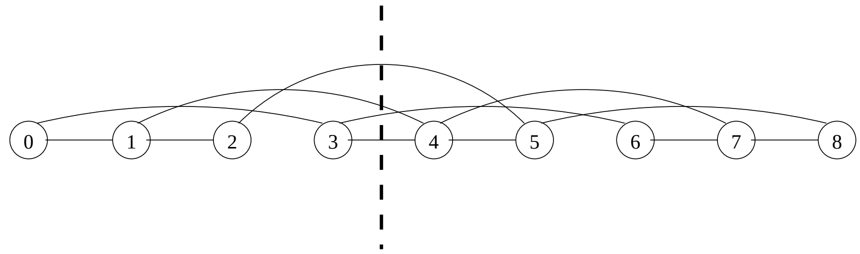

Figure 2: The label–order layout of a 2–dimensional 3–ary array A_3^2 with a *cut* at *gap* 4.

between two nodes assigned to positions $x-1$ and x, $1 \leq x \leq |V(G)|-1$ as *gap* x.

The *cut* of a graph G under a linear layout L at gap x, denoted by $C(G, L, x)$, is the set of edges which connect vertices assigned to positions smaller than x with those in positions greater than or equal to x. That is, $C(G, L, x) = \{(u, v) \in E(G) \mid L(u) < x \leq L(v)\}$. Evidently, $C(G, L, x)$ is equivalent to ∂X in the graph G where $X = \{L^{-1}(0), L^{-1}(1), \ldots, L^{-1}(x-1)\}$. Based on this characterization, we want to prove the following inequality,

$$\frac{|C(A_k^d, L, x)|}{x} \geq \frac{2}{k-1} \tag{6}$$

for any L and $1 \leq x \leq (k^d - 1)/2$ where $x = |X|$.

A related graph topology which is used in Nakano's technique as well as in our proof is the *d-dimensional k-ary clique* K_k^d. The topology of a K_k^d resembles A_k^d. It also has k^d nodes labeled by the integers from 0 to $k^d - 1$. Similar to A_k^d, there is an edge between two nodes if and only if k–ary representations of their labels differ in exactly one digit. However, unlike A_k^d, the absolute value of the difference does not have to be exactly 1. The label–order layout of a K_k^d is defined the same way as that of an A_k^d. Figure 3 illustrates a 2–dimensional 4–ary clique K_4^2. Nakano proved the following lemma in [12].

Lemma 1 *(Nakano, 1993) For the d-dimensional k-ary clique K_k^d, the inequality*

$$|C(K_k^d, I, x)| \leq |C(K_k^d, L, x)|$$

holds for any layout L and gap x.

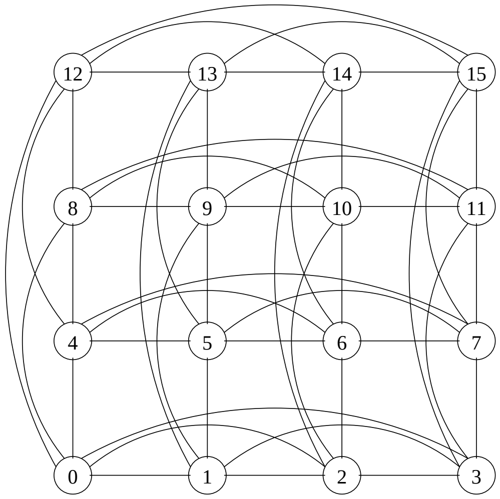

Figure 3: The 2–dimensional 4–ary clique K_4^2.

In other words, the vertices corresponding to the first x numbers in the label–order layout constitute an extremal set in a d–dimensional k–ary clique K_k^d, minimizing the edge–boundary.

Next we give a lemma which characterizes the isoperimetric number problem for A_k^d in terms of a cut width problem for K_k^d when k is odd.

Lemma 2 *Given a d–dimensional k–ary array A_k^d with odd k and an integer x with $1 \leq x \leq (k^d-1)/2$,*

$$i(A_k^d) \geq \frac{2}{k-1} \qquad \textit{if} \quad |C(K_k^d, I, x)| \geq \frac{k+1}{2}x.$$

Proof The embedding of K_k^d into A_k^d is done in the obvious manner: The edge from $u = (a_1, \ldots, a_{i-1}, r, a_{i+1}, \ldots, a_d)$ to $v = (a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_d)$ of the clique with $r < s$ is embedded through the path

$$(a_1, \ldots, a_{i-1}, r, a_{i+1}, \ldots, a_d) \to (a_1, \ldots, a_{i-1}, r+1, a_{i+1}, \ldots, a_d) \to \cdots$$

$$\to (a_1, \ldots, a_{i-1}, s-1, a_{i+1}, \ldots, a_d) \to (a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_d)$$

in the array. Then at most $(k^2-1)/4$ edges of K_k^d are routed through any edge of A_k^d [12]. Consider the set of edges in a cut of A_k^d under a linear layout L at gap x, i.e. $C(A_k^d, L, x)$. There is a set of edges in the K_k^d under the same layout L and gap x, $C(K_k^d, L, x)$, corresponding to this cut as a result of the embedding. This means

$$|C(A_k^d, L, x)| \geq \frac{|C(K_k^d, L, x)|}{\frac{(k^2-1)}{4}}.$$

Thus we have

$$\frac{|C(A_k^d, L, x)|}{x} \geq \frac{|C(K_k^d, L, x)|}{\frac{(k^2-1)}{4}x} \geq \frac{|C(K_k^d, I, x)|}{\frac{(k^2-1)}{4}x}$$

by Nakano's lemma. Hence it suffices to show for $1 \le x \le (k^d - 1)/2$,

$$\frac{|C(K_k^d, I, x)|}{\frac{(k^2-1)}{4} x} \ge \frac{2}{k-1}$$

which is equivalent to showing $|C(K_k^d, I, x)| \ge x(k+1)/2$ for $1 \le x \le (k^d - 1)/2$, and lemma 2 follows. $\square$

Thus if we prove the following claim, then proposition 2, and consequently Theorem 1 follows.

Claim 1 *For k odd, $|C(K_k^d, I, x)| \ge x(k+1)/2$ holds whenever $1 \le x \le (k^d - 1)/2$.*

The proof of claim 1 is by induction on d which we give next. Consider the set X of first x nodes of a K_k^d in label–order layout where $1 \le x \le (k^d - 1)/2$. Let $B_d(x) = |C(K_k^d, I, x)|$. That is, $B_d(x)$ is the number of edges in the layout that connect one of these x nodes to a node in the remaining $k^d - x$. Write $x = a_{d-1}k^{d-1} + \cdots + a_1 k + a_0$ with $0 \le a_i < k$. Thus $a_{d-1}, \ldots, a_1, a_0$ are the digits of x in base k and $x = (a_{d-1} \cdots a_1 a_0)_k$. Note that X is made up of a_{d-1} copies of K_k^{d-1}, a_{d-2} copies of K_k^{d-2} and so on. An example of this is shown in Figure 4. The vertices in dark constitute subset X and the boundary edges ∂X $(= C(K_3^2, I, 4))$ are shown by dashed lines. The subset X consists of one copy of K_3^1 (bottom row vertices) and one copy of K_3^0. We record

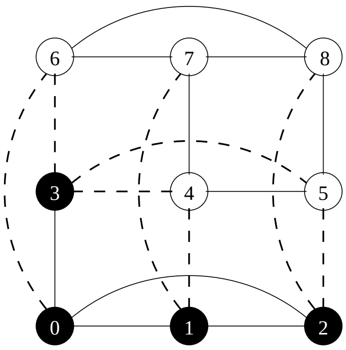

Figure 4: The K_3^2 with set $X = \{0, 1, 2, 3\}$ and $B_2(4) = 8$.

the following observation as a lemma, since we use it repeatedly in the proof.

Lemma 3 *If $x = (a_{d-1} \cdots a_1 a_0)_k$ and $1 \le x \le (k^d - 1)/2$, then $a_{d-1} \le (k-1)/2$.*

Lemma 4 *Suppose $x = (a_{d-1} \cdots a_1 a_0)_k$. Then $B_d(x)$ satisfies the recursion*

$$B_d(x) = a_{d-1}(1 + a_{d-1})k^{d-1} - x(1 + 2a_{d-1} - k) + B_{d-1}(x - a_{d-1}k^{d-1}) \tag{7}$$

with $B_1(a_0) = a_0(k - a_0)$.

Proof Let $a = a_{d-1}$. We show that $B_d(x)$ satisfies the recursion

$$B_d(x) = ak^{d-1}(k-a) - (x - ak^{d-1})a + (x - ak^{d-1})(k - a - 1) + B_{d-1}(x - ak^{d-1}), \tag{8}$$

which can then be simplified to (7). The first term of (8) is the number of edges on the boundary if X consisted of only a copies of K_k^{d-1}. From this we take out the number of edges that link these copies to the remaining portion of X (which we think of as lying in the $(a+1)$-st copy of K_k^{d-1} in K_k^d) and add in turn the edges from this lower dimensional set of cardinality $x - ak^{d-1}$ to the nodes in the remaining $k-a-1$ copies of K_k^d. At this point we have counted all the boundary edges in one dimension, hence we can get rid of this dimension altogether and consider only the boundary edges of the remaining $x - ak^{d-1}$ vertices that reside on the $(a+1)$-st subclique. But the number of boundary edges in that subclique is precisely $B_{d-1}(x - ak^{d-1})$. Hence by adding this we get (8). □

Lemma 5 *Suppose k is odd and $x = (a_{d-1}a_{d-2}\cdots a_0)_k$ with $1 \le x \le (k^d-1)/2$. Then $B_d(x) \ge x(k+1)/2$.*

Proof This lemma is a restatement of claim 1. We prove it by induction on d. For the base case $d=1$, $B_1(a_0) = a_0(k-a_0) \ge x(k+1)/2$ since $x = a_0$. We assume $B_d(x) \ge x(k+1)/2$ whenever $1 \le x \le (k^d-1)/2$. We are required to prove,

$$B_{d+1}(x) = a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(x - a_dk^d) \ge x(k+1)/2 \tag{9}$$

for $1 \le x \le (k^{d+1}-1)/2$ where $x = (a_da_{d-1}\cdots a_0)_k$. We prove this inequality by considering the two possible cases: $x - a_dk^d \le (k^d-1)/2$ and $x - a_dk^d \ge (k^d+1)/2$.

Case I: $x - a_dk^d \le (k^d-1)/2$ In this case the inductive hypothesis is directly applicable and we have

$$a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(x-a_dk^d) \ge a_d(1+a_d)k^d - x(1+2a_d-k) + (x-a_dk^d)(k+1)/2.$$

It suffices to prove

$$a_d(1+a_d)k^d - x(1+2a_d-k) - a_dk^d(k+1)/2 \ge 0.$$

After factoring and rearranging the terms, this is equivalent to

$$\Big(2x - a_dk^d\Big)\Big(a_d - \frac{k-1}{2}\Big) \ge 0. \tag{10}$$

The first factor in (10) is always positive since $x \ge 1$ and a_dk^d is only one of the summands in the base k expansion of x. The second factor is nonnegative by lemma 3.

Note that equality in (10) is possible only for $a_d = (k-1)/2$.

Case II: $x - a_dk^d \ge (k^d+1)/2$ In this case the inductive hypothesis cannot be used directly. Fortunately, however $B_d(x) = B_d(k^d - x)$, and the inductive step can be used as $B_d(k^d - x') \ge (k^d - x')(k+1)/2$, for $x' \ge (k^d+1)/2$. Now the inequality we want to prove becomes

$$a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(k^d - (x - a_dk^d)) \ge x(k+1)/2.$$

By using the inductive hypothesis we have

$$a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(k^d-(x-a_dk^d)) \ge a_d(1+a_d)k^d - x(1+2a_d-k) + (k^d-(x-a_dk^d))(k+1)/2.$$

Thus, after rearrangement of the terms above, it suffices to show

$$k^d\Big(a_d + \frac{k+1}{2}\Big) \geq 2x. \tag{11}$$

Finally we prove inequality (11) in two subcases. First for $a_d = (k-1)/2$, the left–hand side becomes k^{d+1}, whereas the right–hand side is at most $k^{d+1} - 1$ since $x \leq (k^{d+1} - 1)/2$. Now assume $a_d \leq (k-3)/2$. The inequality (11) is equivalent to

$$k^d(k+1)/2 \geq a_d k^d + 2a_{d-1}k^{d-1} + 2a_{d-2}k^{d-2} + \cdots + 2a_0.$$

But $a_{d-1}k^{d-1} + a_{d-2}k^{d-2} + \cdots + a_0 \leq k^d - 1$, since this is a base k expansion. Thus it suffices to prove

$$k^d(k+1)/2 \geq k^d(k-3)/2 + 2k^d - 2.$$

After expansion of terms, this inequality is seen to be equivalent to

$$k^{d+1} + k^d \geq k^{d+1} + k^d - 4.$$

This proves the lemma. Furthermore, if we trace the sequence of inequalities in the proof of Case II in lemma 5, we see that the inequality (9) is always strict in this case. □

This completes the proof of the claim 1. Therefore proposition 2, and consequently Theorem 1 are proved.

4 Isoperimetric Sets and Their Cardinalities

Theorem 2 *The cardinalities of the isoperimetric sets of a A_k^d are*

$$\tfrac{1}{2}(k^d - 1),\ \tfrac{1}{2}(k^d - k),\ \ldots\ ,\ \tfrac{1}{2}(k^d - k^{d-1})$$

for odd k, and $k^d/2$ for even k.

Proof For odd k, an isoperimetric set X of an A_k^d with x vertices must satisfy $B_d(x) = x(k+1)/2$ where $1 \leq x \leq |V(A_k^d)|/2$. From the proof of lemma 5, the equation $B_d(x) = x(k+1)/2$ has exactly d roots in this interval given by

$$(m, 0, 0, \ldots, 0),\ (m, m, 0, \ldots, 0), \ldots, (m, m, \ldots, m)$$

written in base k with $m = (k-1)/2$. These can be written as

$$\tfrac{1}{2}(k^d - 1),\ \tfrac{1}{2}(k^d - k),\ \ldots\ ,\ \tfrac{1}{2}(k^d - k^{d-1}).$$

For even k, any isoperimetric set must make the two sides of inequality (5) equal, which occurs only for $x = k^d/2$. □

Because of the structural symmetry of arrays, there are multiple isoperimetric sets with the same cardinality. For instance, an isoperimetric set, when k is even, is the set of vertices with the first $k^d/2$ smallest labels, i.e. $\{0, 1, \ldots, \frac{k^d}{2} - 1\}$. For odd k the configuration of the isoperimetric sets is more interesting in that they form a nice recursive structure. For instance, the set of vertices with the first $\frac{1}{2}(k^d - k^{d-1})$ smallest labels constitute an isoperimetric set. We can obtain another isoperimetric set if we add to this set the next $\frac{(k-1)}{2}k^{d-2}$ vertices. Continuing this way, we can obtain an isoperimetric set corresponding to any one of the d cardinalities listed above.

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