The Edge-isoperimetric Number of Generalized Cylinders

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Abstract

A d–dimensional generalized cylinder G^d is the Cartesian product of d graphs, each of which is a path graph or a cycle graph. In this paper, we use an embedding technique to show that the edge–isoperimetric number $i(G^d)$ is given by

$$i(G^d) = \min\{i(P_k), i(C_l)\}$$

where P_k and C_l are the largest path graph and the largest cycle graph among the factors in G^d, and the factor on the right achieving the minimum value has an even number of vertices. As a byproduct, we obtain the isoperimetric number of multidimensional even tori (product of cycle graphs on even number of vertices) and multidimensional even arrays (product of path graphs on even number of vertices). In fact for tori and arrays, our result only requires the size of the largest factor to be even. We also obtain a complete description of the corresponding isoperimetric sets as well as exact formulas for the bisection width of these generalized cylinders.

Keywords: Generalized cylinder, product graph, array, torus, isoperimetric number, bisection width, edge separator.

1 Introduction

Given a graph G and a subset X of its vertices, let ∂X denote the *edge–boundary* of X, i.e. the set of edges which connect vertices in X with vertices in $V(G) \setminus X$. The *edge-isoperimetric number*, or simply the *isoperimetric number*, of G is defined as

$$i(G) = \min_{1 \le |X| \le \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|} \ . \tag{1}$$

That is, the set of vertices of G is partitioned into two nonempty sets and the ratio of the number of edges between the two parts and the number of vertices in the smaller one is minimized. As examples of isoperimetric numbers:

- $i(K_k) = \lceil \frac{k}{2} \rceil$ for the complete graph K_k with k vertices,
- $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ for the path graph P_k with k vertices,

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- $i(C_k) = 2/\lfloor \frac{k}{2} \rfloor$ for the cycle graph C_k with k vertices.

A subset X which achieves the minimum ratio in (1) is called an *isoperimetric set.* We refer the reader to Mohar [13] or Chung [8] for a discussion of basic results and various interesting properties of $i(G)$ and to Bezrukov [5] for a comprehensive survey of this and related problems.

In this paper, we are interested in the isoperimetric properties of *generalized cylinders* which are the Cartesian product of cycles and paths. The Cartesian product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent if and only if u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. The constituent graphs G and H are called *factors.* A *generalized d–dimensional cylinder*, or simply a *cylinder* is a d-fold product $G^d = G_{k_1} \times G_{k_2} \times \cdots \times G_{k_d}$ where each G_{k_i} is either a path P_{k_i} or a cycle C_{k_i} with k_i vertices. Figure 1 illustrates the 2–dimensional cylinder $P_4 \times C_3$. A *multidimensional*

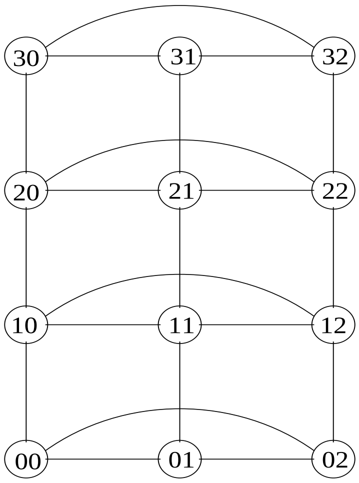

Figure 1: The 2–dimensional cylinder $P_4 \times C_3$.

array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ and a *multidimensional torus* $C^d = C_{k_1} \times C_{k_2} \times \cdots \times C_{k_d}$ are special cases of generalized cylinders.

In this paper we prove that for a generalized cylinder $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, $k_1 \geq k_i$, $l_1 \geq l_i$, and k_1, l_1 both even (in fact under slightly weaker assumptions on the parities of k_1 and l_1 as given in Theorem 1) we have

$$i(G^d) = \min\{i(P_{k_1}), i(C_{l_1})\} = \min\{\frac{2}{k_1}, \frac{4}{l_1}\}. \tag{2}$$

As a special case, we obtain that the formula

$$i(G^d) = \min_j\{i(G_{k_j})\} \tag{3}$$

for the isoperimetric number holds for multidimensional arrays and multidimensional tori whenever the largest factor has even parity.

Theorem 1 *Given a cylinder G^d,*

- *if G^d is a multidimensional array, i.e. $G^d = P_{k_1} \times \cdots \times P_{k_d}$ with $k_1 \geq k_i$ and k_1 even,*

$$i(G^d) = i(P_{k_1}) = \frac{2}{k_1} \tag{4}$$

- *if G^d is a multidimensional torus, i.e. $G^d = C_{l_1} \times \cdots \times C_{l_d}$ with $l_1 \geq l_i$ and l_1 even,*

$$i(G^d) = i(C_{l_1}) = \frac{4}{l_1} \tag{5}$$

- *if $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, $k_1 \geq k_i$, $l_1 \geq l_i$,*

$$i(G^d) = \min\{i(P_{k_1}), i(C_{l_1})\} = \begin{cases} \frac{2}{k_1} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ \frac{4}{l_1} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases} \tag{6}$$

- *The cardinality of the isoperimetric sets of G^d in equations (4),(5) and (6) is $|V(G^d)|/2$.*

We also give a description of these isoperimetric sets. The following corollary regarding *the bisection width $bw(G^d)$* of G^d is another consequence of Theorem 1 . The notion of bisection width and its relationship with the notion of isoperimetric number will be explained in the next section.

Corollary 1 *Given a cylinder G^d with n vertices,*

- *if G^d is a multidimensional array, i.e. $G^d = P_{k_1} \times \cdots \times P_{k_d}$ with $k_1 \geq k_i$ and k_1 even,*

$$bw(G^d) = \frac{n}{k_1}$$

- *if G^d is a multidimensional torus, i.e. $G^d = C_{l_1} \times \cdots \times C_{l_d}$ with $l_1 \geq l_i$ and l_1 even,*

$$bw(G^d) = 2\frac{n}{l_1}$$

- *if $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, $k_1 \geq k_i$, then*

$$bw(G^d) = \begin{cases} \frac{n}{k_1} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ 2\frac{n}{l_1} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases}$$

1.1 Motivation

The notion of the isoperimetric number of a graph G serves as a measure of connectivity of G as it quantifies the minimal interaction between a set of vertices X and its complement $V(G) \setminus X$ in terms of the number of edges between them. This idea is also important in algorithm design. For instance, the notion of isoperimetric number is implicit in the divide-and-conquer strategy in graph algorithms. To illustrate, consider an algorithm which adopts divide-and-conquer strategy where the set of vertices of the underlying graph is split into two near-equal size parts such that the algorithm can be run on the two corresponding subgraphs recursively, and the results are combined to get the solution for the original problem. The combining of results at the last step needs to be carried out with minimal effort if such a scheme is expected to be efficient. It is desirable to split the graph in such a way as to keep the interaction between the two partitions (in terms of the number of edges connecting them) as small as possible.

The isoperimetric number is also closely related to the notion of the *bisection width*, $bw(G)$, of a graph G. This is the minimum number of edges that must be removed from G in order to split $V(G)$ into two *equal–sized* (within one) subsets. If known, one can use the isoperimetric number of a graph G to establish a lower bound for its bisection width using the fact that

$$bw(G) \geq i(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor . \tag{7}$$

In fact, the lower bounds which yield the exact expressions for the bisection width in Corollary 1 are obtained directly from inequality (7) and Theorem 1.

1.2 Outline

The outline of the rest of this paper is as follows. In section 2 we give a summary of previous work on isoperimetric properties of various families of product graphs. The proof of our main result appears in section 3. In section 4 we give the cardinality and the description of the isoperimetric sets of a cylinder, as well as the subsets achieving the bisection width. In section 5, we give a conclusion and future research considerations.

2 A summary of previous work

There has been a significant amount of research in the area of isoperimetric bounds in various popular classes of graphs such as arrays and tori. The notion of isoperimetric number of a graph is related to the theory of extremal sets in graphs. An *extremal set* of a graph for a given m is, in a broad sense, a configuration of m vertices with

i) minimum number of boundary edges, or

ii) maximum number of spanned edges

among all m–vertex subsets. The problem of finding extremal sets of the first (or, second) type is called *the minimum–boundary–edge problem* (or, *the maximum–induced–edge problem*). It can be shown that the minimum–boundary–edge and the maximum–induced–edge problems are equivalent for regular graphs [7].

The maximum–induced–edge problem for the hypercube (hence the minimum–boundary–edge problem, because of its regularity) was solved by Harper [10] and extended by Lindsey [12] to the d–dimensional k–ary clique. In both instances, there is a nested structure of solutions, and the first m vertices in *lexicographical order* constitute an extremal set. The maximum–induced–edge problem for the d–dimensional k–ary array A_k^d was first solved by Bollobás and Leader [7]. Their result was later extended to multidimensional arrays by Ahlswede and Bezrukov [1] who also gave a solution for $P_{k_1} \times P_{k_2}$ for the minimum–boundary–edge problem. The first nontrivial bounds on the minimum–boundary–edge problem for the d–dimensional k–ary arrays were given by Bollobás and Leader [7].

A natural approach for the exact calculation of $i(A_k^d)$ is to try to exploit the recursive Cartesian product structure $A_k^d = P_k \times A_k^{d-1}$. For instance, Mohar [13] gave a proof for $i(P_{k_1} \times P_{k_2}) =$

$\min\{i(P_{k_1}), i(P_{k_2})\}$. It can be proved that in general $i(P_k \times G) = i(P_k)$ where G is any connected graph with k vertices [2]. However, it was shown in [2] that $i(P_k \times G) \neq i(P_k)$ if G has more than k vertices even when $i(G) \geq i(P_k)$ (it is already known that $i(P_k \times G) < i(P_k)$ if $i(G) < i(P_k)$). Azizoğlu and Eğecioğlu [3] proved that $i(A_k^d) = i(P_k)$ using an inductive technique and also gave a description of the isoperimetric sets. For general product graphs, Chung and Tetali [9] proved that the isoperimetric number of an arbitrary Cartesian product graph is within a factor $\frac{1}{2}$ of the smallest isoperimetric number of its factors.

3 The isoperimetric number of a cylinder

We shall focus on proving equation (6) of Theorem 1, since equations (4) and (5) are proved using essentially the same technique. Specifically, we will prove the following.

Proposition 1 *Given a cylinder $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ with $d = r + s$; $k_1 \geq k_i$, $l_1 \geq l_i$,*

$$i(G^d) = \min\{i(P_{k_1}), i(C_{l_1})\} = \begin{cases} \frac{2}{k_1} & \text{if } 2k_1 \geq l_1 \text{ and } k_1 \text{ even,} \\ \frac{4}{l_1} & \text{if } l_1 \geq 2k_1 \text{ and } l_1 \text{ even.} \end{cases} \tag{8}$$

Proof We prove this by showing that the expressions on the right–hand side are both lower and upper bounds for $i(G^d)$. We use the following well-known result to prove the upper bound.

$$i(G_1 \times G_2 \times \cdots \times G_r) \leq \min\{i(G_1), i(G_2), \ldots, i(G_r)\} \tag{9}$$

See Chung [8] for a proof of (9). Clearly, $i(P_{k_1}) = \min_i\{i(P_{k_i})\}$ and $i(C_{l_1}) = \min_i\{i(C_{l_i})\}$. If $2k_1 \geq l_1$ then $i(P_{k_1}) \leq i(C_{l_1})$ and for even k_1 we have,

$$i(G^d) \leq \frac{2}{k_1} = i(P_{k_1}) = \min\{i(P_{k_1}), \ldots, i(P_{k_r}), i(C_{l_1}), \ldots, i(C_{l_s})\}. \tag{10}$$

Similarly, if $l_1 \geq 2k_1$ and l_1 is even then,

$$i(G^d) \leq \frac{4}{l_1} = i(C_{l_1}) = \min\{i(P_{k_1}), \ldots, i(P_{k_r}), i(C_{l_1}), \ldots, i(C_{l_s})\}, \tag{11}$$

since $i(C_{l_1}) \leq i(P_{k_1})$.

To prove the lower bound, we extend an embedding technique which Leighton used in [11] to obtain a lower bound for the bisection width of arrays. Specifically, we embed a directed complete graph K_n with n vertices into G^d where n is the number of vertices of G^d. Any partition of vertices in K_n induces a partition in G^d as a result of the embedding. Using this, we argue that the number of boundary edges in a partition of vertices in G^d cannot be less than that of K_n divided by the *congestion* of the embedding. The congestion is the maximum number of edges of K_n routed through any edge of G^d.

Before we describe our embedding, consider a nonempty subset X of vertices of G^d with $|X| \leq n/2$. Let X' be the subset of vertices of the complete graph associated with the vertices in X as a result of the embedding. Note that $|\partial X'| = 2|X'|(n - |X'|)$ and $|\partial X| \geq |\partial X'|/c$ where c is the congestion of the embedding. Therefore

$$\frac{|\partial X|}{|X|} \geq \frac{|\partial X'|}{c|X|} \geq \frac{2|X'|(n - |X'|)}{c|X|}$$

Since $|X| = |X'|$ and $1 \leq |X| \leq n/2$,

$$\frac{2|X'|(n-|X'|)}{c|X|} = \frac{2(n-|X'|)}{c} \geq \frac{2\frac{n}{2}}{c} = \frac{n}{c} \ . \tag{12}$$

Thus we have,

$$i(G^d) \geq \frac{n}{c} \tag{13}$$

The embedding we describe next has congestion c with

$$c = \begin{cases} n\frac{k_1}{2} & \textit{if } 2k_1 \geq l_1 \textit{ and } k_1 \textit{ even}, \\ n\frac{l_1}{4} & \textit{if } l_1 \geq 2k_1 \textit{ and } l_1 \textit{ even}. \end{cases}$$

which, when combined with (13), yields the desired lower bound. □

The details of the embedding are given next.

3.1 Embedding a directed complete graph into a cylinder

Given a generalized cylinder $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ where $d = r + s$, for notational convenience let $k_{r+i} := l_i$ for $1 \leq i \leq s$. We identify each vertex v by a label $(v_1, \ldots, v_d)$ where $0 \leq v_i \leq k_i - 1$ for $1 \leq i \leq d$ as shown in Figure 1. Note that there are a total of $n = k_1 k_2 \cdots k_d$ vertices.

We embed into G^d a directed complete graph K_n with n vertices where the vertices are identified with the vertices of G^d. The edge from node $u = (u_1, \ldots, u_d)$ to node $v = (v_1, \ldots, v_d)$ of K_n is embedded by using a left–to–right dimensional routing scheme. That is, first the value u_1 is "corrected" into v_1, then u_2 into v_2, and so on, until all u_i have been corrected.

When correcting u_i into v_i for $1 \leq i \leq r$, (i.e. when the factor in the i^{th} dimension of the cylinder is a path) the correction is done by taking the (unique) shortest path between u_i and v_i. In other words, u_i is incremented (or decremented) until it becomes equal to v_i. If $r+1 \leq i \leq r+s$, since the i^{th} factor is a cycle, there are exactly two shortest paths between u_i and v_i when they are diametrically opposite *and* k_i is even. For instance, take $k_i = 6$, $u_i = 2$ and $v_i = 5$ as shown in Figure 2. The shortest paths between vertices 2 and 5 are 2–3–4–5 and 2–1–0–5. In all other cases the shortest path is unique. To avoid ambiguity when there are multiple shortest paths, we always take the one with increasing label values (transition from $k_i - 1$ to 0 is assumed to be increasing, whereas 0 to $k_i - 1$ is decreasing.) Thus, in our example, the path 2–3–4–5 is taken as shown in Figure 2. We next prove the congestion lemma.

Lemma 1 *Let c be the congestion of the embedding of an n–vertex directed complete graph K_n into a cylinder $G^d = P_{k_1} \times \cdots \times P_{k_r} \times C_{l_1} \times \cdots \times C_{l_s}$ with n vertices, as described above, where $d = r + s$, $k_1 \geq k_i$ and $l_1 \geq l_i$. Then we have,*

$$c = \begin{cases} n\frac{k_1}{2} & \textit{if } 2k_1 \geq l_1 \textit{ and } k_1 \textit{ even}, \\ n\frac{l_1}{4} & \textit{if } l_1 \geq 2k_1 \textit{ and } l_1 \textit{ even}. \end{cases}$$

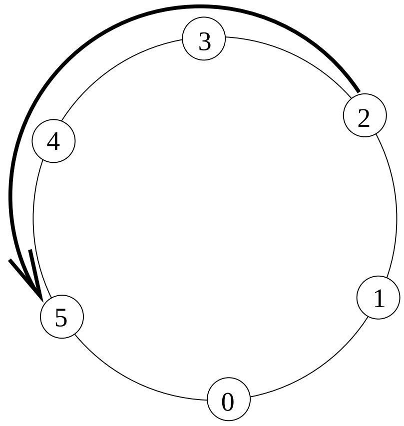

Figure 2: The cycle with 6 vertices and embedding of edge $\langle 2,5\rangle$.

Proof For convenience, we again take $k_{r+i} = l_i$ for $1 \le i \le s$. Define c_i for $1 \le i \le d$ to be the maximum number of edges of K_n embedded in any edge *in the* i^{th} *dimension of* G^d. It suffices to show, for $1 \le i \le r$,

$$c_i = \begin{cases} n\frac{k_i}{2} & \textit{if } k_i \textit{ is even,} \\ n\frac{k_i^2-1}{2k_i} & \textit{if } k_i \textit{ is odd.} \end{cases}$$

and for $r+1 \le i \le r+s$,

$$c_i = \begin{cases} n\frac{k_i}{4} & \textit{if } k_i \textit{ is even,} \\ n\frac{k_i^2-1}{4k_i} & \textit{if } k_i \textit{ is odd,} \end{cases}$$

since, if these hold, then

$$c = \max_{1\le i\le d}\{c_i\} = \begin{cases} n\frac{k_1}{2} & \textit{if } 2k_1 \ge l_1 \textit{ and } k_1 \textit{ even,} \\ n\frac{l_1}{4} & \textit{if } l_1 \ge 2k_1 \textit{ and } l_1 \textit{ even} \end{cases}$$

as desired.

Let $e = \langle (e_1,\ldots,e_i,\ldots,e_d),(e_1,\ldots,e_i+1,\ldots,e_d)\rangle$ be an edge of G^d in the i^{th} dimension. Also, let $u = (u_1,\ldots,u_d)$ and $v = (v_1,\ldots,v_d)$ be two distinct nodes of the complete graph for which the embedding results in the directed edge from u to v to be embedded through e. Note that c_i is the maximum number of such (u,v) pairs. Since routing is done in a left-to-right fashion, we must have

$$\begin{array}{ccc} u_{i+1} = e_{i+1} & , & v_1 = e_1 \\ u_{i+2} = e_{i+2} & , & v_2 = e_2 \\ \vdots & & \vdots \\ u_d = e_d & , & v_{i-1} = e_{i-1} \end{array}$$

along with either $u_i \le e_i$ and $v_i \ge e_i+1$ or $v_i \le e_i$ and $u_i \ge e_i+1$ depending on the direction of the embedded edge, i.e. $\langle u,v\rangle$ or $\langle v,u\rangle$. This leaves $k_1\cdots k_{i-1}k_{i+1}\cdots k_d$ choices for $u_1,\ldots,u_{i-1},v_{i+1},\ldots,v_d$.

The number of choices for u_i and v_i depends on whether the i^{th} factor of G^d is a path or cycle. If it is a path then u_i and v_i can be chosen one of

$$(e_i+1)(k_i-(e_i+1))+(k_i-(e_i+1))(e_i+1)=2(e_i+1)(k_i-e_i-1)$$

different ways. Since,

$$2(e_i+1)(k_i-e_i-1)\leq\begin{cases}\frac{k_i^2}{2} & \textit{if } k_i \textit{ is even},\\ \frac{k_i^2-1}{2} & \textit{if } k_i \textit{ is odd}\end{cases}$$

we have, for $1\leq i\leq r$,

$$c_i=\max_{e_i}\left\{k_1\cdots k_{i-1}k_{i+1}\cdots k_d\Big(2(e_i+1)(k_i-e_i-1)\Big)\right\}=\begin{cases}n\frac{k_i}{2} & \textit{if } k_i \textit{ is even},\\ n\frac{k_i^2-1}{2k_i} & \textit{if } k_i \textit{ is odd}.\end{cases}$$

as desired. If the i^{th} factor of G^d is a cycle then the number of ways u_i and v_i can be chosen is exactly

$$\frac{\frac{k_i}{2}\left(\frac{k_i}{2}+1\right)}{2}+\frac{\frac{k_i}{2}\left(\frac{k_i}{2}-1\right)}{2}=\frac{k_i^2}{4}$$

when k_i is even, and

$$\frac{\frac{k_i-1}{2}\frac{k_i+1}{2}}{2}+\frac{\frac{k_i-1}{2}\frac{k_i+1}{2}}{2}=\frac{k_i^2-1}{4}$$

when k_i is odd. Thus, for $r+1\leq i\leq d$, we have

$$c_i=\begin{cases}n\frac{k_i}{4} & \textit{if } k_i \textit{ is even},\\ n\frac{k_i^2-1}{4k_i} & \textit{if } k_i \textit{ is odd}\end{cases}$$

as needed. □

The proof of our main Proposition 1 now follows by combining the congestion lemma with the inequality (13), which gives the lower bound $i(G^d)\geq\min\{i(P_k),i(C_l)\}$, and the two inequalities (10), and (11) which give the reverse inequality $i(G^d)\leq\min\{i(P_k),i(C_l)\}$.

4 Isoperimetric sets, their cardinalities and the bisection width

Proposition 2 *The cardinality of the isoperimetric sets of G^d in equation (8) is $|V(G^d)|/2$.*

Proof An isoperimetric set must make the two sides of inequality (12) equal, which can happen only if it has $|V(G^d)|/2$ vertices. □

Even though any isoperimetric set must have cardinality $|V(G^d)|/2$ vertices, because of the structural symmetry of cylinders, there may be multiple isoperimetric sets. For $G^d=P_{k_1}\times\cdots\times P_{k_r}\times C_{l_1}\times\cdots\times C_{l_s}$ where $d=r+s$, the set X given below is an isoperimetric set.

$$X=\begin{cases}\{u=(u_1,\ldots,u_d)\mid u_1<\frac{k_1}{2}\} & \textit{if } 2k_1\geq l_1 \textit{ and } k_1 \textit{ even},\\ \{u=(u_1,\ldots,u_d)\mid u_{r+1}<\frac{l_1}{2}\} & \textit{if } l_1\geq 2k_1 \textit{ and } l_1 \textit{ even}.\end{cases}$$

It can be seen that the partition $(X,\overline{X})$ is indeed a bisection achieving the bisection width of G^d given in Corollary 1.

5 Summary and future considerations

We have used an embedding technique to calculate the isoperimetric number of a large subclass of generalized cylinders. This class includes multidimensional tori and multidimensional arrays in which the largest factor has even parity. Unfortunately, our embedding technique does not work when the largest factor in the product has an odd number of vertices. It would be desirable to show that for any generalized cylinder $G^d = G_{k_1} \times G_{k_2} \times \cdots \times G_{k_d}$,

$$i(G^d) = \min_j \{i(G_{k_j})\}$$

regardless of the parities of the factors involved. It appears that instead of an embedding, a technique similar to the one used in [3] may work in this general case. This is under investigation.

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