A Bijection for the Convolution of Central Binomial Coefficients

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Abstract

Let $A_n = \binom{2n}{n}$ for $n \geq 0$ denote the the n-th element in the axis of symmetry of the Pascal triangle. The generating function for A_n is $(1-4t)^{-1/2}$, from which it follows that $A_0A_n + A_1A_{n-1} + \cdots + A_nA_0 = 4^n$. Here I describe a bijective proof of this identity based on lattice paths.

1 Introduction

Alastair King and Andrew Swann of the University of Bath asked for a combinatorial (counting) proof of the convolution identity

$$\sum_{i=0}^{n} A_i A_{n-i} = A_0A_n + A_1A_{n-1} + \cdots + A_nA_0 = 4^n \tag{1}$$

where $A_n = \binom{2n}{n}$ for $n \geq 0$ are the central binomial coefficients [4]. King and Swann were intrigued after Richard Cleyton found an application of this identity in quaternionic geometry in his doctoral dissertation [2]. Peter Cameron brought the combinatorial problem to my attention by posting it on the Internet [1].

Since the generating function for A_n is $(1-4t)^{-1/2}$, this identity is immediate by algebraic means. What is required is a "counting" proof of (1).

2 Preliminaries

Given a word w and a letter a, denote by $|w|$ the length of w, and by $|w|_a$ the number of occurrences of a in w. The empty word is denoted by ϵ. If

$w = uv$, then u is a *prefix* of w. A *Dyck word* w is a word over the alphabet $\Sigma = \{0, 1\}$ with the following properties:

1. For each prefix u of w, $|u|_1 \geq |u|_0$,
2. $|w|_1 = |w|_0$.

Evidently Dyck words are of even length. Let D denote the collection of Dyck words. If binary words are interpreted as lattice paths with diagonal steps ($1 = \nearrow, 0 = \searrow$), then D corresponds to those paths (Dyck paths) from the origin to some point $(2n, 0)$, that stay weakly above the x-axis. The number of Dyck paths of length $2n$ is the Catalan number $C_n = \binom{2n}{n}/(n+1)$ [5].

Each word w over Σ^{2n} with $|w|_0 = |w|_1 = n$ can be uniquely factored (using an extra letter z as marker) as $w' = d_0 z d_1 z d_2 \cdots d_{k-1} z d_k$ where k is even and each $d_i \in D$. These are called *Catalan* words [6]. The first half of the z's in w' correspond to 0 in w, and the second half to 1. Consequently, there are $A_n = \binom{2n}{n}$ Catalan words of length $2n$. Identity (1) is equivalent to showing that each word $w \in \Sigma^{2n}$ corresponds bijectively to a pair of Catalan words (u, v) of total length $2n$.

An arbitrary word w over Σ^{2n} has a similar factorization as w', and can be unambiguously recovered from w' as soon as we specify the number r of z's from the left that correspond to 0 in w. The pair (w', r) is the *Catalan factorization* of w. This is best illustrated by shining a light source to the lattice path of w from the left and from the right. The steps corresponding to 0's that are lit from the left give the first r occurrences of z in w', those 1's that are lit from the right are the remaining z's in w'. This is illustrated in Figure 1 for $w = 00101100010010111010$. For a Catalan word w' the number $|w'|_z$ is even, and $r = \frac{1}{2}|w'|_z$, so this information is encoded in w' and need not be specified. Given a Catalan word u of length $2k$, let $\phi_0(u)$,

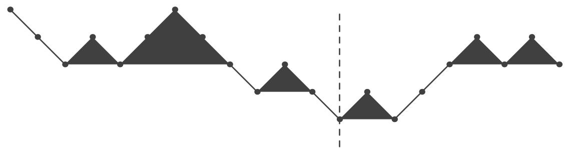

Figure 1: Catalan factorization: The lattice path and the factorization for $w = 00101100010010111010 \rightarrow w' = zz101100z10z10zz1010$, $r = 4$.

(resp. $\phi_1(u)$) denote the word in Σ^{2k} obtained by substituting 0 (resp. 1) for each z in u. For an ordered pair of Catalan words (u, v), it is possible to interpret the z's in u as 0, and the z's in v as 1 and identify u with $\phi_0(u)$, and v with $\phi_1(v)$.

3 The Bijection

The idea of the bijection is to break the Catalan factorization (w', r) of w into two pieces as u and v by splitting it immediately after the r-th z, as indicated by the vertical line in Figure 1. There is a difficulty when r is an odd number, in which case a type of transposition encodes this fact.

Start with a pair of Catalan words (u, v) where $|u| = 2k$ and $|v| = 2(n-k)$ for some k. From this pair, we construct $w \in \Sigma^{2n}$ as follows:

1. If $u = \epsilon$, then $w = \phi_1(v)$.
2. If $u = \alpha z$ then $w = \phi_0(u)\phi_1(v)$,
3. If $u = \alpha d$ for a Dyck word $d = 1\beta 0$ (where $\alpha = \epsilon$ or $\alpha = \gamma z$), then $w = \phi_0(\alpha\beta)\,(01)\,\phi_1(v)$.

Below are examples for $n = 10$ illustrating these three cases.

Example 1: Given pair $u = \epsilon$, $v = zz101100z10z10zz1010 \rightarrow w =$ 11101100110110111010.

Example 2: Given pair $u = zz101100z10z$, $v = 10zz1010 \rightarrow w =$ (001011000100)(10111010) (this is the word whose path is displayed in Figure 1).

Example 3: Given pair $u = z10z110110001100$, $v = 1010$ (with $\alpha = z10z$, $d = 1\beta 0$, and $\beta = 1011000110$) $\rightarrow w =$ (01001011000110)(01)(1010). The indicated segment of the path of $\phi_0(u)\phi_1(v)$ on top portion of Figure 2 is transposed by a left circular rotation to obtain w as shown in the bottom.

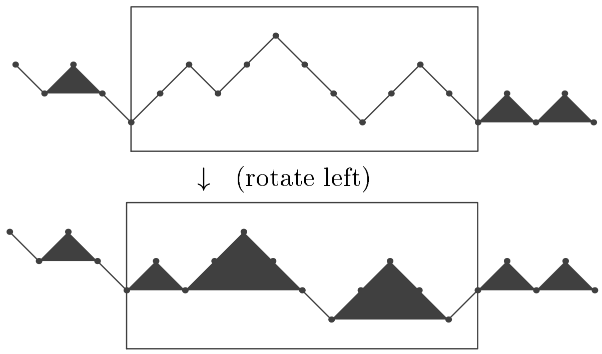

Figure 2: From (u, v) to w.

To go in the other direction, consider the Catalan factorization (w', r) of an arbitrary $w \in \Sigma^{2n}$. Write $w' = u'v'$ where the last letter of u' is the r-th

z from the left in w. If r is odd, then there is a factorization $w' = xd_1zd_2zy$ where d_1 and d_2 are the Dyck words immediately before and after the r-th z in w'.

1'. If $r = 0$, then $u = \epsilon$, $v = v'$.

2'. If $r > 0$ is even, then $u = u'$, $v = v'$,

3'. If r is odd and $w' = xd_1zd_2zy$, then $u = x1d_10d_2$, $v = y$.

The three examples below for $n = 10$ illustrate these cases.

Example 1': Given word $w = 10110011110010101101$. Catalan factorization of w is $w' = 101100zz11001010z10z$, $r = 0$, which is mapped to $u = \epsilon$, $v = 101100zz11001010z10z$, with $(\phi_0(u), \phi_1(v)) = (\epsilon, 10110011110010101101)$.

Example 2': Given word $w = 10010110001100101101$. Catalan factorization of w is $w' = 10z101100z110010z10z$, $r = 2$, which is mapped to $u = 10z101100z$, $v = 110010z10z$, with $(\phi_0(u), \phi_1(v)) = (1001011000, 1100101101)$.

Example 3': Given word $w = 10010010010110110110$. Catalan factorization of w is $w' = 10z10z10z10z10z10z10$, $r = 3$. Locating the transposed portion of u as marked on top of Figure 3, we obtain the second path in the figure, which is then mapped to $u = 10z10z110010$, $v = 10z10z10$, with $(\phi_0(u), \phi_1(v)) = (100100110010, 10110110)$.

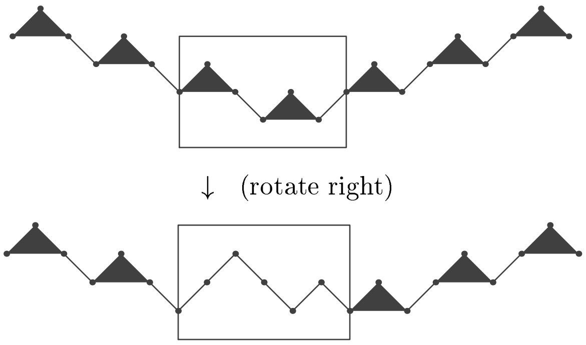

Figure 3: From w to (u, v).

4 Examples and Remarks

The table in Figure 4 gives the complete set of binary words of length 4 and the corresponding pairs (u, v) produced by the bijection.

w	$(u\ ,\ v)$	$(\phi_0(u)\ ,\ \phi_1(v))$	w	$(u\ ,\ v)$	$(\phi_0(u)\ ,\ \phi_1(v))$
1111	$(\epsilon\ ,\ zzzz)$	$(\epsilon\ ,\ 0011)$	0111	$(10\ ,\ zz)$	$(10\ ,\ 01)$
1110	$(\epsilon\ ,\ zz10)$	$(\epsilon\ ,\ 0110)$	0110	$(10\ ,\ 10)$	$(10\ ,\ 10)$
1101	$(\epsilon\ ,\ z10z)$	$(\epsilon\ ,\ 0101)$	0101	$(1010\ ,\ \epsilon)$	$(1010\ ,\ \epsilon)$
1100	$(\epsilon\ ,\ 1101)$	$(\epsilon\ ,\ 1100)$	0100	$(z10z\ ,\ \epsilon)$	$(0101\ ,\ \epsilon)$
1011	$(\epsilon\ ,\ 10zz)$	$(\epsilon\ ,\ 1001)$	0011	$(zz\ ,\ zz)$	$(01\ ,\ 01)$
1010	$(\epsilon\ ,\ 1010)$	$(\epsilon\ ,\ 1010)$	0010	$(zz\ ,\ 10)$	$(01\ ,\ 10)$
1001	$(1100\ ,\ \epsilon)$	$(1100\ ,\ \epsilon)$	0001	$(zz10\ ,\ \epsilon)$	$(0110\ ,\ \epsilon)$
1000	$(10zz\ ,\ \epsilon)$	$(1001\ ,\ \epsilon)$	0000	$(zzzz\ ,\ \epsilon)$	$(0011\ ,\ \epsilon)$

Figure 4: Binary words w of length 4 and the corresponding pairs (u, v).

Remark: For any real number a we have the expansion

$$(1-at)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} (\frac{a}{4})^n\, t^n$$

so that the coefficients have a combinatorial meaning when $a = 4r$ for some positive integer r. In this case

$$\sum_{i=0}^{n} \binom{2i}{i} r^i \binom{2(n-i)}{n-i} r^{n-i} = 4^n r^n,$$

which is equivalent to (1). Therefore (1) is the only distinct identity of its kind.

Remark: Alastair King suggested an even nicer application of Catalan factorization of Section 2 by interpreting the right hand side of identity (1) as the number of $w \in \Sigma^{2n+1}$ with $|w|_0$ odd [3]. Even though this is a somewhat skewed interpretation of 4^n, it does simplify the bijective step through the following observation: The Catalan factorization of such a w has an odd number of z's, and contains a unique $z = z'$ such that

1. There are an even number of z's before z', all of which correspond to 0 in w,
2. There are an even number of z's after z', all of which correspond to 1 in w.

This determines a pair of Catalan words (u, v), as in Section 3. To go backwards from a pair of Catalan words (u, v) to $w \in \Sigma^{2n+1}$ with an odd number of 0's, set $w = \phi_0(u)z'\phi_1(v)$ where $z' = 0$ if $|u|_0 + |v|_0$ is even, and $z' = 1$ otherwise.

References

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