Extremal Sets Minimizing Dimension–normalized Boundary in Hamming Graphs

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Abstract

We prove that the set of first k vertices of a Hamming graph in reverse–lexicographic order constitutes an extremal set minimizing the dimension–normalized edge–boundary over all k–vertex subsets of the graph. This generalizes a result of Lindsey and can be used to prove a tight lower bound for the isoperimetric number and the bisection width of arrays.

Keywords: Hamming graph, product graph, array, isoperimetric number, bisection width, extremal set, partition, Schur-convexity, majorization.

1 Introduction

In this paper, we consider questions of the following general form: Given a graph G and a natural number k, what is the optimum value of a certain quantity in a set of k vertices of G? The desired quantity could be the number of edges between a set of k vertices and its complement (i.e. the size of the boundary), or the number of edges induced by a set of k vertices, etc. The sets achieving the optimum value are called *extremal sets.*

Specifically, we study extremal sets in Hamming graphs minimizing the size of the edge–boundary of a set of vertices of given size, where boundary edges along each dimension are *normalized* by a weight determined by that dimension as shall soon be explained.

We introduce some notation and terminology. Given a graph G and a subset X of its vertices, let ∂X denote the *edge–boundary*, or simply *boundary*, of X. This is the set of edges connecting vertices in X with vertices not in X (i.e. the complement of X). A *d–dimensional Hamming graph* H^d is a graph with $k_1 \times k_2 \times \cdots \times k_d$ vertices, $k_1 \leq k_2 \leq \cdots \leq k_d$, each having a unique label $l = \langle l_1, l_2, \cdots, l_d \rangle$ where $0 \leq l_i \leq k_i - 1$. There is an edge between two vertices iff their labels differ in exactly one digit. A *d–dimensional array* A^d resembles H^d with the exception that two vertices are adjacent if and only if their labels differ in exactly one digit *and* the difference is

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exactly one. Examples of a 2–dimensional Hamming graph and a 2–dimensional array are shown in Figure 1.

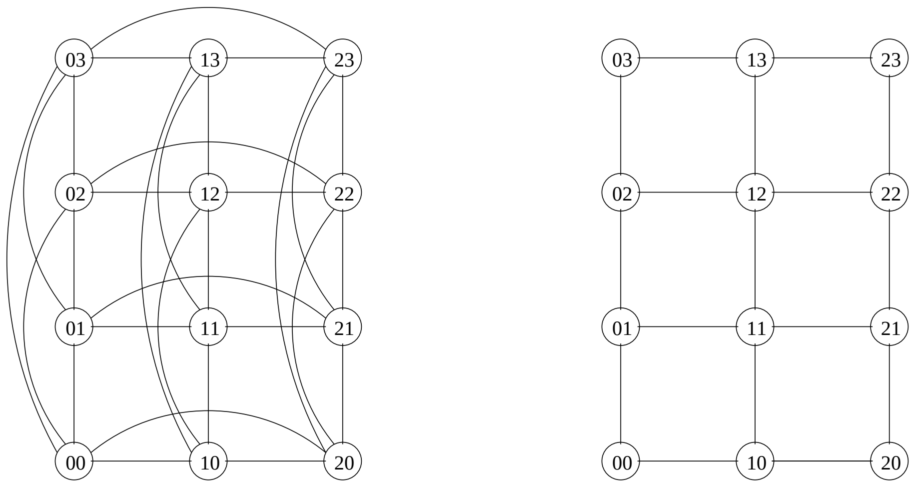

Figure 1: The 2–dimensional Hamming graph $K_3 \times K_4$ and array $P_3 \times P_4$.

The *Cartesian product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent iff u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. The constituent graphs G and H are called *factors*. A Hamming graph can be characterized as the Cartesian product of a number of *complete graphs* of different sizes, i.e. $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ where K_r is a complete graph on r vertices. Similarly, A^d can be characterized as the Cartesian product of a number of *path graphs* of varying length, i.e. $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ where P_r is a path graph (chain) with r vertices.

Lindsey [19] proved that the set of first k vertices of a Hamming graph in *lexicographic order* constitutes an extremal set minimizing the boundary ∂X over all k–element subsets X. The lexicographic order is defined as follows: In the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ with $k_1 \leq k_2 \leq \cdots \leq k_d$, vertex $x = \langle x_1, \cdots, x_d \rangle$ precedes vertex $y = \langle y_1, \cdots, y_d \rangle$ in lexicographic order iff there exists an index i such that $x_1 = y_1$, $x_2 = y_2$, $\ldots$, $x_{i-1} = y_{i-1}$ and $x_i < y_i$ holds. Intuitively, in lexicographic order, we traverse the Hamming graph in the direction of the *next largest factor* starting with the vertex labeled $\langle 0, 0, \cdots, 0 \rangle$. For instance, the vertices of the Hamming graph in Figure 1 in lexicographic order are labeled

$$00,\ 01,\ 02,\ 03,\ 10,\ 11,\ 12,\ 13,\ 20,\ 21,\ 22,\ 23.$$

Our aim in this paper is to determine and describe extremal sets of Hamming graphs minimizing the dimension–normalized boundary. This is defined next.

Definition 1 *Given a Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ and a subset X of its vertices,* the dimension–normalized boundary *$B(X)$ of X is defined as*

$$B(X) = \frac{|\partial_1 X|}{c_1} + \frac{|\partial_2 X|}{c_2} + \cdots + \frac{|\partial_d X|}{c_d} \qquad (1)$$

where for $1 \le i \le d$, $\partial_i X$ is the set of boundary edges along dimension i and

$$c_i = \begin{cases} k_i^2 & \text{if } k_i \text{ is even,} \\ k_i^2 - 1 & \text{if } k_i \text{ is odd.} \end{cases} \tag{2}$$

Specifically, we prove that the set of first k vertices in *reverse–lexicographic order* constitutes an extremal set minimizing the dimension–normalized boundary over all k–element subsets in a Hamming graph. The definition of the reverse–lexicographic order is similar to that of the lexicographic order: In the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ with $k_1 \le k_2 \le \cdots \le k_d$, vertex $x = \langle x_1, \cdots, x_d \rangle$ precedes vertex $y = \langle y_1, \cdots, y_d \rangle$ in reverse–lexicographic order iff there exists an index i such that $x_d = y_d$, $x_{d-1} = y_{d-1}$, $\ldots$, $x_{i+1} = y_{i+1}$ and $x_i < y_i$ holds. In other words, we move in the direction of the *next smallest factor* starting at the vertex labeled $\langle 0, 0, \cdots, 0 \rangle$. To illustrate, the vertices of the Hamming graph in the above example listed in reverse–lexicographic order are

00, 10, 20, 01, 11, 21, 02, 12, 22, 03, 13, 23.

We should point out that there are other sets of vertices which are structurally equivalent to the sets specified in our definitions of lexicographic or reverse–lexicographic orders. These are obtained by the symmetries of the underlying graph. For instance, in Figure 1, another ordering structurally equivalent to the lexicographic ordering would be 23, 22, 21, 20, 13 etc.. Similarly, the sets defined by the initial segments of the ordering 23, 13, 03, 22, 12, etc., give rise to sets structurally identical to those in reverse–lexicographic order.

We state our claim formally in the following theorem.

Theorem 1 *Given a d–dimensional Hamming graph H^d, let X be any k–vertex subset of $V(H^d)$ and $\overline{X}$ be the set of first k vertices of H^d in reverse–lexicographic order. Then $B(\overline{X}) \le B(X)$.*

Interestingly, when all factors of H^d have equal size, the lexicographic and reverse–lexicographic orders both result in structurally symmetric subsets and hence are equivalent with respect to extremal sets minimizing the boundary (dimension–normalized or otherwise). Therefore Theorem 1 is trivially true when $k_1 = k_2 = \cdots = k_d$ by Lindsey's result, since the denominators c_i in (1) will all be equal and minimizing $B(X)$ will be equivalent to minimizing $|\partial_1 X| + |\partial_2 X| + \cdots + |\partial_d X| = |\partial X|$, i.e. the size of the boundary of X.

In the next section, we describe the notion of the isoperimetric number, which is a quantity closely related to extremal sets. The isoperimetric number problem for special classes of graphs provides the basis of our motivation for this work.

1.1 Motivation

An important quantity in the theory of graphs is the *isoperimetric number* $i(G)$ of a graph G, defined as

$$i(G) = \min_{1 \le |X| \le \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|} \tag{3}$$

where $X \subseteq V(G)$. That is, the set of vertices of G is partitioned into two nonempty sets and the ratio of the number of edges between the two parts and the number of vertices in the smaller one is minimized. A subset X in (3) achieving the equality is called an *isoperimetric set.*

The notion of the isoperimetric number of a graph G serves as a measure of connectivity of G as it quantifies the minimal interaction between a set of vertices X and its complement $V(G) \setminus X$ in terms of the number of edges between them. In many instances, the isoperimetric number of a graph can be used to obtain a tight lower bound for its *bisection width* as well. We refer the reader to Mohar [22] or Chung [12] for a discussion of basic results and various interesting properties of $i(G)$.

At present, the isoperimetric number of an array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ is known only when either $k_1 = k_2 = \cdots = k_d$ (see Azizoğlu and Eğecioğlu [4]), or the size of the largest factor is even (see Azizoğlu and Eğecioğlu [5]. The latter is also implicit in [10] ([17])). The techniques used to obtain these results seem to fail in the general case. However using the notion of extremal sets minimizing dimension–normalized boundary, one can show that

$$i(A^d) = \min_i \frac{1}{\lfloor \frac{k_i}{2} \rfloor}$$

The technique used involves embedding a Hamming graph into A^d and associating these extremal sets with isoperimetric sets of the array. We refer the reader to [6] for details.

1.2 A summary of previous results

There has been a significant amount of research in the area of isoperimetric bounds on various popular classes of graphs such as Hamming graphs, arrays and tori. We shall only mention those results in this area which pertain to our discussion and refer the reader to Bezrukov [8] for a comprehensive survey and Bollobás [9] for a general discussion of this and related topics.

As mentioned before, an extremal set of a graph for a given k is, in a broad sense, a configuration of k vertices with

- minimum number of boundary edges, or
- maximum number of spanned edges

among all such k–vertex subsets of the given graph. The problem of finding extremal sets of the first (or, second) type is called *the minimum–boundary–edge problem* (or, *the maximum–induced–edge problem*). It can be shown that the minimum–boundary–edge and the maximum–induced–edge problems are equivalent for regular graphs [11]. We remark that one can easily obtain the isoperimetric number of a given graph if the extremal sets of the first type are known (and the boundary is actually computable). Evidently, an extremal set X with $\lfloor |V(G)|/2 \rfloor$ vertices in a given graph G is a bisection of G.

The maximum–induced–edge problem (hence the minimum–boundary–edge problem, because of its regularity) for the hypercube (d–dimensional binary Hamming graph) was solved by Harper [14] and extended by Lindsey [19] to the d–dimensional k–ary Hamming graph. In both instances, there is a nested structure of solutions, and the first k vertices in *lexicographic order* constitute an extremal set. The maximum–induced–edge problem for the d–dimensional k–ary array A_k^d was solved by Bollobás and Leader [11]. Since A_k^d is not regular, this result does not automatically give a solution to the minimum–boundary–edge problem. It was later extended to general arrays by Ahlswede and Bezrukov [1] who also gave a solution for $P_{k_1} \times P_{k_2}$ for the minimum–boundary–edge problem. The first nontrivial bounds on the minimum–boundary–edge problem for the

d–dimensional k–ary arrays are in Bollobás and Leader [11]. Unfortunately, however, the bounds obtained are not tight enough to yield an exact formula for $i(A_k^d)$.

Similar problems have been studied in the literature for the vertex–boundary of a given configuration of vertices. For instance, for the d–dimensional k–ary torus, Bollobás and Leader [10] solved the vertex–boundary problem for even k. Riordan [24] later extended their result by giving an ordering of vertices, on the d–dimensional even torus, which minimizes the number of vertices at shortest distance t from the vertices in the ordering. Wang and Wang [25] solved this problem for $P_\infty \times \cdots \times P_\infty$, i.e. the *$d$–dimensional infinite array*, where the minimum is taken over all nonempty finite subsets of vertices. In their result, each P_∞ may be infinite in both directions or in one direction only. They also gave a simple ordering of the vertices in which the first k vertices constitute an extremal set minimizing the vertex–boundary. In a recent paper, Harper [15] solved the vertex–boundary problem on Hamming graphs.

1.3 Outline

The outline of the remainder of this paper is as follows. In Section 2 we consider the case of two–dimensional Hamming graphs. First we define the terminology we use and state a number of basic facts on restricted integer partitions, majorization, and Schur-convexity. Then we identify potential extremal sets in H^2 as integer partitions inside a rectangle. The problem of showing that the set of first k vertices of H^2 in reverse–lexicographic order constitutes an extremal set minimizing the dimension–normalized edge–boundary over all k–vertex subsets, becomes the problem of maximization of a certain function on partitions, which is a linear combination of two Schur-convex functions. However the function itself is not Schur-convex, and the identification of the partition on which the maximum is achieved is actually done using an inductive argument. The main result of this section is Lemma 1. In Section 3 we extend the proof to the higher–dimensional case. This is done by induction on the dimension, using the two–dimensional result as the base case. Finally, concluding remarks are given in Section 4.

2 The two–dimensional case

Let $H^2 = K_m \times K_n$ be a given 2–dimensional Hamming graph. Without loss of generality, we may assume that $m = k_1 \leq k_2 = n$. Consider a subset X of vertices in H^2. Let X' be the subset of vertices of H^2 obtained by pushing (compressing) all the vertices in X as far downward, and then to the left in H^2 as possible. It is easy to see (and proved in [19], [16]) that $B(X') \leq B(X)$ since the number of boundary edges in either dimension will not increase as a result of this procedure. A subset X' in the compressed form corresponds to a *partition* of the integer $|X|$ contained in the $m \times n$ rectangle.

We give below the definitions and properties of partitions that we will use in our proof of Theorem 1. The reader is referred to [2] for further details.

Partitions

A *partition* λ of an integer N is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers (called *parts*) satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = N$. We put $|\lambda| = N$. The Ferrers diagram of λ is a 2–dimensional array of unit cells (or nodes) in which row i from the bottom has λ_i cells, and the rows are left justified. It is clear that in our case, $\lambda = X'$ forms a partition of $|X|$ whose Ferrers diagram is contained in the $m \times n$ rectangle, i.e. $\lambda_i \leq m$ for $1 \leq i \leq \ell$ (i.e. each part at most m), and $\ell \leq n$ (i.e. number of parts at most n). We use $\mathbb{P}(m,n)$ to denote the set of these partitions. Thus we may assume that an extremal set is a partition $\lambda \in \mathbb{P}(m,n)$, and use the symbol $\mathbb{P}(m,n)$ to refer to $H^2 = K_m \times K_n$ when we are not interested in the graph structure of H^2, but just the placement of the subset λ. We may augment partitions by adding parts of zero length and write $\sum_{i \geq 1} \lambda_i$ for $|\lambda|$. We also identify partitions with their diagrams when there is no confusion.

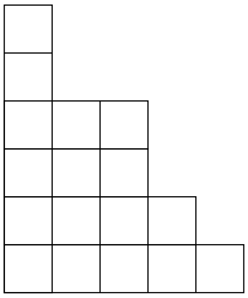

Figure 2: The partition $\lambda = (5,4,3,3,1,1)$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ by choosing λ'_i as the number of parts of λ that are $\geq i$. The partition λ' is called the *conjugate* of λ. Geometrically, λ' is obtained from λ by reflection in the main diagonal (equivalently by counting the cells in successive columns of λ). For example, the conjugate of $(5,4,3,3,1,1)$ is $(6,4,4,2,1)$. Clearly $|\lambda| = |\lambda'|$, and if $\lambda \in \mathbb{P}(m,n)$ then $\lambda' \in \mathbb{P}(n,m)$.

Majorization, Schur-convexity, and transfer

Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ of N, λ is *majorized* by μ, written $\lambda \prec \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1 + \mu_2 + \cdots + \mu_k, \qquad k = 1, 2, \ldots, N.$$

Majorization is also referred to as the *dominance* or *natural* order ([20], Ch. 1). As soon as $N \geq 6$, majorization is not a total ordering. For example the partitions $(3,1,1,1)$, and $(2,2,2)$ of 6 are not comparable. However reverse–lexicographic ordering on partitions is a linear extension of $\prec$. Thus

$$\lambda \prec \mu \;\Rightarrow\; \lambda \leq \mu.$$

Furthermore $\lambda \prec \mu \;\Leftrightarrow\; \mu' \prec \lambda'$ (see [20], (1.11)). A real valued function g defined on partitions of an integer N is said to be *Schur–convex* (see [21], Ch. 3) if

$$\lambda \prec \mu \;\Rightarrow\; g(\lambda) \leq g(\mu).$$

We make use of the following special case of a result of Schur, 1923; Hardy, Littlewood, and Polya, 1929 (see [21], Ch 3, Proposition C.1)

Proposition 1 *Suppose ϕ is a real-valued convex function on $\mathbb{R}$ and N is a positive integer. Then the function*

$$g(\lambda) = \sum_{i\geq 1} \phi(\lambda_i)$$

is Schur-convex on partitions of N.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ with $\mu_i > \mu_j$, the transformation that takes μ to $\rho = (\rho_1, \rho_2, \ldots, \rho_N)$ defined by

$$\begin{aligned} \rho_i &= \mu_i - 1 \\ \rho_j &= \mu_j + 1 \\ \rho_k &= \mu_k, \qquad\qquad k \neq i, j, \end{aligned}$$

is called a *transfer* from i to j. By a result of Muirhead, if $\lambda \prec \mu$, then λ can be derived from μ by successive application of a finite number of transfers ([21], Ch. 5, D.1; [20], (1.16)).

Now consider a partition $\lambda \in \mathbb{P}(m,n)$ where $k_1 = m \leq n = k_2$, which corresponds to a compressed set in $H^2 = K_m \times K_n$. Let $\partial_m \lambda$ and $\partial_n \lambda$ be sets of horizontal and vertical boundary edges of λ, respectively. Then we have

$$|\partial_m \lambda| = \sum_{\lambda_i > 0} \lambda_i (m - \lambda_i) \qquad \textit{and} \qquad |\partial_n \lambda| = \sum_{\lambda'_j > 0} \lambda'_j (n - \lambda'_j)$$

After substituting these in (1) and eliminating constant terms, we see that finding a subset $\lambda \in K_m \times K_n$ minimizing $B(\lambda)$ is equivalent to *maximizing* the following function f

$$\begin{aligned} f(\lambda) &= c_1 \sum_{i=1}^{\lambda'_1} \lambda_i^2 + c_2 \sum_{j=1}^{\lambda_1} \lambda_j'^2 \\ &= \gamma_n \sum_{i\geq 1} \lambda_i^2 + \gamma_m \sum_{j \geq 1} \lambda_j'^2 \end{aligned} \tag{4}$$

on $\mathbb{P}(m,n)$ $(m \leq n)$, where

$$\gamma_n = \begin{cases} n^2 & \textit{if } n \textit{ is even,} \\ n^2 - 1 & \textit{if } n \textit{ is odd} \end{cases}$$

in accordance with the definition of the weights c_i in (2).

Note that the function f is not Schur-convex. In other words, transfer operators ([21], Ch. 5, D.1) or equivalently raising/lowering operators (see [20], (1.15-1.16)) which move from a given λ to a smaller one in the linear order while keeping the value of f nondecreasing are insufficient to prove this fact[1]. As an example take $m = 4$, $n = 8$, $\lambda = (4,1,1,1,1,1,1)$, $\mu = (4,2,1,1,1,1)$. Then $\lambda \prec \mu$ by a single transfer as shown in Figure 3, but $2240 = f(\lambda) > f(\mu) = 2208$.

[1] Transfer arguments can be used to prove Lemma 1 in the special case when $n \geq m^2$.

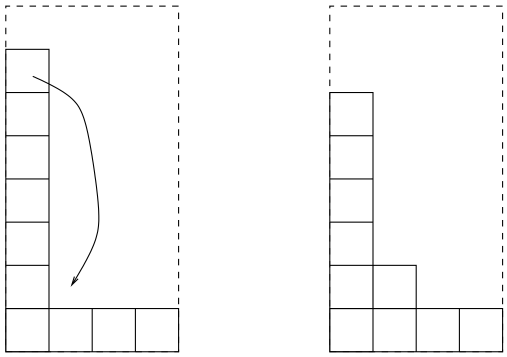

Figure 3: μ is obtained from λ by single transfer (lowering the indicated cell). Here $m = 4$, $n = 8$, $\lambda \prec \mu$ but $f(\lambda) = 2240$, whereas $f(\mu) = 2208$.

2.1 Extremal sets for the 2–dimensional Hamming graph

Now we are ready to prove the 2–dimensional case which is stated using the terminology of Hamming graphs in the following lemma.

Lemma 1 *Given a 2–dimensional Hamming graph $H^2 = K_m \times K_n$ with $m \leq n$, let λ be any k–vertex subset of $V(H^2)$ and $\overline{\lambda}$ be the set of first k vertices of H^2 in reverse–lexicographic order. Then $f(\overline{\lambda}) \geq f(\lambda)$. That is*

$$\gamma_n \sum_{i\geq 1} \overline{\lambda}_i^2 + \gamma_m \sum_{j\geq 1} \overline{\lambda}_j'^2 \geq \gamma_n \sum_{i\geq 1} \lambda_i^2 + \gamma_m \sum_{j\geq 1} \lambda_j'^2 \tag{5}$$

Proof We only give the proof for n and m both even. The other cases are similar. After rearranging inequality (5), we are required to prove

$$n^2 \sum_{i\geq 1} (\overline{\lambda}_i^2 - \lambda_i^2) \geq m^2 \sum_{j\geq 1} (\lambda_j'^2 - \overline{\lambda}_j'^2) \tag{6}$$

We shall prove the claim by induction on n and m for any given $m \leq n$. The base of the induction is the case where $n = m$, which is already known to be true by Lindsey's result as remarked earlier.

The proof of the inductive step involves different cases based on the number of elements of λ which are in the topmost and bottommost rows of H^2. We call the i^{th} row (from the bottom) of H^2 'empty' with respect to a partition λ if no vertices in this row is a member of λ, and 'nonempty' otherwise. In terms of parts of λ these two conditions mean $\lambda_i = 0$ and $\lambda_i > 0$, respectively. Likewise, the i^{th} row of H^2 is called 'complete' if all vertices in this row belong to the partition (i.e. $\lambda_i = m$), and 'incomplete' (i.e. $\lambda_i < m$) otherwise. According to this terminology, the proof is divided into the following three cases with respect to λ, covering all possible configurations:

Case (1): empty top row (i.e. $\lambda_n = 0$),
Case (2): complete bottom row (i.e. $\lambda_1 = m$),

Case (3): nonempty top and incomplete bottom rows (i.e. $\lambda_n > 0$ and $\lambda_1 < m$).

Each of these will be proved next.

Case (1): *Empty top row*

Now we assume that the claim is true for $n-1$ where $n-1 \geq m$. Since n is even, $n-1$ is odd, and hence $\gamma_{n-1} = (n-1)^2 - 1 = n^2 - 2n$ according to Definition 1. Thus by induction hypothesis we have

$$(n^2 - 2n) \sum_{i=1}^{n-1} (\overline{\lambda}_i^2 - \lambda_i^2) \geq m^2 \sum_{j \geq 1} (\lambda_j'^2 - \overline{\lambda}_j'^2) \tag{7}$$

In this case, the partition λ fits into the 2–dimensional Hamming graph $K_m \times K_{n-1}$ obtained by discarding the top row of H^2, i.e. $\lambda \in \mathbb{P}(m, n)$. Thus the induction hypothesis is applicable and combining inequality (6) and (7), it suffices to prove

$$n^2 \sum_{i \geq 1} (\overline{\lambda}_i^2 - \lambda_i^2) \geq (n^2 - 2n) \sum_{i \geq 1} (\overline{\lambda}_i^2 - \lambda_i^2) \tag{8}$$

since $\lambda_n = \overline{\lambda}_n = 0$. After cancellation of terms, this reduces to proving

$$\sum_{i \geq 1} \overline{\lambda}_i^2 \geq \sum_{i \geq 1} \lambda_i^2.$$

Since $|\lambda| = |\overline{\lambda}|$ and the function $\phi(x) = x^2$ is convex on $\mathbb{R}$, this inequality follows from the majorization result of Proposition 1 .

Note that an alternate proof of the last inequality can be given by the following observation: By combining pairs of rows of λ into a single row if possible (if not, then make one row with size m and another row having the leftover elements) while maintaining the partition property, we eventually convert λ into $\overline{\lambda}$ and do not decrease the function $\sum_{i \geq 1} \lambda_i^2$ at any intermediate step.

Case (2): *Complete bottom row*

Consider H^2 and the given configuration λ residing in H^2 with a complete bottommost row. By removing this row from H^2 we obtain a 2–dimensional Hamming graph $K_m \times K_{n-1}$ and a new partition λ^* containing all rows of λ but the bottommost one with $\lambda^* \in \mathbb{P}(m, n)$. Let $\overline{\lambda}^*$ be the set of first $|\lambda^*|$ vertices of $K_m \times K_{n-1}$ in reverse–lexicographic order. Note that the first $|\lambda|$ vertices of $K_m \times K_n$ in reverse–lexicographic order are the bottommost row of m elements, followed by $\overline{\lambda}^*$.

Now by induction hypothesis, we have $f(\overline{\lambda}^*) \geq f(\lambda^*)$, which means

$$(n^2 - 2n) \sum_{i \geq 2} (\overline{\lambda}_i^2 - \lambda_i^2) \geq m^2 \sum_{j \geq 1} (\lambda_j'^2 - \overline{\lambda}_j'^2)$$

Combining this with inequality (6), it suffices to prove

$$n^2 \sum_{i \geq 1} (\overline{\lambda}_i^2 - \lambda_i^2) \geq (n^2 - 2n) \sum_{i \geq 2} (\overline{\lambda}_i^2 - \lambda_i^2)$$

which is equivalent to

$$n^2 \sum_{i \geq 1} (\overline{\lambda}_i^2 - \lambda_i^2) \geq (n^2 - 2n) \sum_{i \geq 1} (\overline{\lambda}_i^2 - \lambda_i^2)$$

since $\overline{\lambda}_1^2 = \lambda_1^2 = m^2$. But this is the same inequality as (8) which was already proved in the previous case.

Case (3): *Nonempty top and incomplete bottom rows*

In this case, the partition λ resides in the 2–dimensional Hamming graph $K_{m-1} \times K_n$ obtained by removing the rightmost column of H^2. By applying the induction hypothesis to λ in $K_{m-1} \times K_n$, we shall first obtain an intermediate configuration $\lambda^* \in \mathbb{P}(m-1, n)$ such that $f(\lambda^*) \geq f(\lambda)$. Note that, although λ^* is in reverse–lexicographic order with respect to the Hamming graph $K_{m-1} \times K_n$ in which it resides, it is not so with respect to H^2 and, thus, will have to be converted into $\overline{\lambda}$ mentioned in the statement of the theorem where $f(\overline{\lambda}) \geq f(\lambda^*)$.

First, the inequality

$$n^2 \sum_{i \geq 1} (\lambda_i^{*2} - \lambda_i^2) \geq (m^2 - 2m) \sum_{j=1}^{m-1} (\lambda_j'^2 - \lambda_j^{*\prime 2})$$

is obtained by applying the induction hypothesis. Next, to establish $f(\lambda^*) \geq f(\lambda)$, it suffices to prove

$$(m^2 - 2m) \sum_{j=1}^{m-1} (\lambda_j'^2 - \lambda_j^{*\prime 2}) \geq m^2 \sum_{j=1}^{m-1} (\lambda_j'^2 - \lambda_j^{*\prime 2})$$

After cancellation of terms, this reduces to proving

$$\sum_{j \geq 1} \lambda_j^{*\prime 2} \geq \sum_{j \geq 1} \lambda_j'^2$$

which follows from the majorization result as before.

To show $f(\overline{\lambda}) \geq f(\lambda^*)$, λ^* is handled in two subcases:

Subcase (1) where $|\lambda| = |\lambda^*| \leq (m-1)(n-1)$: In this case, the top row of λ^* in H^2 is empty and by Case 1 above, we have $f(\overline{\lambda}) \geq f(\lambda^*)$.

Subcase (2) where $|\lambda| = |\lambda^*| > (m-1)(n-1)$: In this case, the top row is nonempty and the argument for subcase 1 cannot be applied.

Suppose λ^* has x elements in its top row. Note that $0 < x < m$ and the configuration $\hat{\lambda}$ obtained by moving all x elements in the top row of λ^* to the rightmost column in H^2 is a legitimate partition. Using this, we first show that $f(\hat{\lambda}) \geq f(\lambda^*)$. Note that $f(\hat{\lambda})$ now has an empty top row (as well as a complete bottom row) and thus, by Case 1 or Case 2 of the proof, we have $f(\overline{\lambda}) \geq f(\hat{\lambda})$, and hence $f(\overline{\lambda}) \geq f(\lambda^*)$ as desired.

To show that $f(\hat{\lambda}) \geq f(\lambda^*)$, we calculate explicitly the values

$$f(\hat{\lambda}) = n^2 \Big(xm^2 + (n-1-x)(m-1)^2 \Big) + m^2 \Big((m-1)(n-1)^2 + x^2 \Big)$$

and

$$f(\lambda^*) = n^2\Big((n-1)(m-1)^2 + x^2\Big) + m^2\Big(xn^2 + (m-1-x)(n-1)^2\Big).$$

We are required to prove

$$n^2\Big(xm^2 + (n-1-x)(m-1)^2 - (n-1)(m-1)^2 - x^2\Big) \; +$$

$$m^2\Big((m-1)(n-1)^2 + x^2 - xn^2 - (m-1-x)(n-1)^2\Big) \geq 0$$

After cancellation and rearrangement of terms and dividing the resulting expression by $x(n-m)$, this reduces to proving

$$2nm \geq (n+m)(x+1).$$

Since the largest value for x is $x = m-1$, it suffices to prove

$$2nm \geq (n+m)m$$

which is evident for $n \geq m$. □

3 The higher–dimensional case

In this section we prove Theorem 1 for arbitrary dimension d.

Proof The proof is by induction on d with $d = 2$ being the base case. We assume $k_1 \leq k_2 \leq \cdots \leq k_d$ for $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ and vertices are labeled by d-tuples $\langle l_1, l_2, \cdots, l_d \rangle$ where $0 \leq l_i \leq k_i - 1$.

Given an arbitrary point configuration X in H^d, we permute the k_d $(d-1)$–dimensional Hamming subgraphs along dimension d such that successive subgraphs each have fewer members of X. Now we apply the induction hypothesis to each of these subgraphs. Note that these moves cannot increase $B(X)$ since $|\partial_d X|$ cannot increase and by induction hypothesis

$$\frac{|\partial_1 X_i|}{c_1} + \cdots + \frac{|\partial_{d-1} X_i|}{c_{d-1}}$$

is smallest for each subgraph i, where X_i is the set of elements of X that are in subgraph i. This means that candidate extremal sets in higher dimensions are among *higher–dimensional partitions* (see [2], Ch. 11), which are contained in the d-dimensional parallelpiped $k_1 \times k_2 \times \cdots \times k_d$. Now we repeat the same steps for subgraphs along dimension $d-1$ as well.

Consider the $(d-2)$–dimensional Hamming subgraphs of H^d when dimensions d and $d-1$ are fixed. We call such a subgraph 'complete' iff its vertices are completely contained in X, and 'incomplete' otherwise. We shall show that if there are more than one incomplete subgraph then these can be combined without increasing $B(X)$. To this end we give some definitions and develop proper notation.

First, suppose $P^*_{p,q}$ and $P^*_{r,s}$ are sets of vertices of two such incomplete $(d-2)$–dimensional Hamming subgraphs where p,r and q,s are coordinates of dimension d and $d-1$ respectively. Without loss of generality, assume

$$\frac{p}{c_d} + \frac{q}{c_{d-1}} \geq \frac{r}{c_d} + \frac{s}{c_{d-1}} \tag{9}$$

Next, let $P_{p,q} = P^*_{p,q} \cap X$, $P_{r,s} = P^*_{r,s} \cap X$ and $Y = X \setminus (P_{p,q} \cup P_{r,s})$. Now given two disjoint subsets S and T of $V(H^d)$, let

$$B(S,T) = \frac{|\partial_1(S,T)|}{c_1} + \frac{|\partial_2(S,T)|}{c_2} + \cdots + \frac{|\partial_d(S,T)|}{c_d} \tag{10}$$

where c_i is as defined before and $\partial_i(S,T)$ is the set of edges in dimension i having one end in S and the other in T. Note that, in this notation $B(X) = B(X, V(H^d)\setminus X)$. By inclusion–exclusion, the following holds.

$$B(X) = B(Y) + B(P_{p,q}) + B(P_{r,s}) - 2(B(Y,P_{p,q}) + B(Y,P_{r,s})) - 2B(P_{p,q},P_{r,s}) \tag{11}$$

We claim that if as many elements in $P_{r,s}$ as possible are moved to $P^*_{p,q}$ then $B(X)$ does not increase. To this end, consider the terms in equation (11). First we remark that, by virtue of the reverse–lexicographic order, we must have $p \neq r$ and $q \neq s$, and therefore $B(P_{p,q},P_{r,s}) = 0$ in equation (11). Furthermore, because of inequality (9), $B(Y,P_{p,q}) + B(Y,P_{r,s})$ cannot decrease by this move, and $B(Y)$ is constant. Finally, we claim that $B(P_{p,q}) + B(P_{r,s})$ does not increase.

To prove this, note that any vertex $v \in (P_{p,q} \cup P_{r,s})$ is adjacent to $k_d - 1$ and $k_{d-1} - 1$ vertices in dimensions d and $d-1$, respectively. Thus, moving vertices from $P_{r,s}$ to $P^*_{p,q}$ does not change the boundary along dimensions d and $d-1$. Therefore, it suffices to prove

$$B'(P_{p,q}) + B'(P_{r,s}) \geq B'(P'_{p,q}) + B'(P'_{r,s}) \tag{12}$$

where

$$B'(X) = \frac{|\partial_1 X|}{c_1} + \cdots + \frac{|\partial_{d-2} X|}{c_{d-2}}$$

and $P'_{p,q}$ and $P'_{r,s}$ are the new subsets corresponding to $P_{p,q}$ and $P_{r,s}$ respectively, after elements are moved from $P_{r,s}$ to $P^*_{p,q}$.

To prove inequality (12), first suppose that all of $P_{r,s}$ fits in the complement of $P_{p,q}$ with respect to $P^*_{p,q}$. Thus we can place elements of $P_{r,s}$ into $P^*_{p,q} \setminus P_{p,q}$ in a set structurally identical to the one given by reverse–lexicographic order, i.e. starting with vertex $\langle k_1 - 1, k_2 - 1, \cdots, k_{d-2} - 1, q, p\rangle$ of H^d and expanding in the direction of the smallest factor of the Hamming graph. That is

$$\langle k_1 - 1, k_2 - 1, \cdots, k_{d-3} - 1, k_{d-2} - 1, q, p\rangle \to \langle k_1 - 1, k_2 - 1, \cdots, k_{d-3} - 1, k_{d-2} - 2, q, p\rangle \to \dots$$

$$\dots \to \langle k_1 - 1, k_2 - 1, \cdots, k_{d-3} - 2, k_{d-2} - 1, q, p\rangle \to \langle k_1 - 1, k_2 - 1, \cdots, k_{d-3} - 2, k_{d-2} - 2, q, p\rangle \to \dots$$

and so on. In this case, we have $B'(P'_{r,s}) = 0$ since $P'_{r,s} = \phi$ and $B'(P'_{p,q})$ can be written as $B'(P'_{p,q}) = B'(P_{p,q}) + B'(P_{r,s}) - 2B'(P_{p,q},P_{r,s})$. Substituting these values in inequality (12), it suffices to prove

$$B'(P_{p,q}) + B'(P_{r,s}) \geq B'(P_{p,q}) + B'(P_{r,s}) - 2B'(P_{p,q},P_{r,s})$$

which obviously holds since $B'(P_{p,q},P_{r,s}) \geq 0$. We remark that $P'_{p,q}$ is not in reverse–lexicographic order at this point since it consists of two subsets each of which is structurally in reverse–lexicographic order. Nevertheless, by an easy application of the induction hypothesis, we can convert it to the reverse–lexicographic order without increasing $B'(P'_{p,q})$.

Now assume that not all elements of $P_{r,s}$ fit into $P^*_{p,q}$. Take first $|P^*_{p,q} \setminus P_{p,q}|$ vertices in reverse–lexicographic order in $P^*_{r,s}$. These vertices are in $P_{r,s}$. Call this set of vertices Y_2 and set $Y_1 = P_{r,s} \setminus Y_2$. After moving all vertices in Y_2 to $P^*_{p,q}$, we put Y_1 in reverse–lexicographic order $\overline{Y}_1$ within $P^*_{r,s}$. Then, inequality (12) reduces to proving

$$B'(P_{p,q}) + B'(P_{r,s}) \geq B'(Y_1)$$

as $B'(Y_1) \geq B'(\overline{Y_1})$ holds by the induction hypothesis. Now note that $B'(P_{r,s}) = B'(Y_1) + B'(Y_2) - 2B'(Y_1, Y_2)$ and $B'(Y_2) = B'(P_{p,q})$ since Y_2 and $P_{p,q}$ are complementary in $P^*_{p,q}$. Thus the above inequality is equivalent to

$$B'(Y_2) \geq B'(Y_1, Y_2)$$

which obviously holds. Thus $B(P_{p,q}) + B(P_{r,s})$ does not increase as claimed. By applying this process to all $(d-2)$–dimensional incomplete subgraphs, we can assume that X has only one incomplete $(d-2)$–dimensional Hamming subgraph.

Finally we treat the $(d-2)$–dimensional Hamming subgraphs as single vertices and use the 2–dimensional case to minimize $B(X)$ by putting them in reverse–lexicographic order with the only incomplete one highest in the order. This completes the proof of Theorem 1. □

4 Conclusions

We proved that the set of first k vertices of the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ $(k_1 \leq k_2 \leq \cdots \leq k_d)$, in reverse–lexicographic order constitutes an extremal set minimizing the dimension–normalized edge–boundary over all k–vertex subsets of the graph. The boundary edges $\partial_i X$ along the i-th dimension of $X \subset V(H^d)$ are normalized by a weight

$$c_i = \begin{cases} k_i^2 & \text{if } k_i \text{ is even,} \\ k_i^2 - 1 & \text{if } k_i \text{ is odd,} \end{cases}$$

which naturally arises in the isoperimetric number problem for d-dimensional arrays. The weighted boundary to be minimized is then

$$B(X) = \frac{|\partial_1 X|}{c_1} + \frac{|\partial_2 X|}{c_2} + \cdots + \frac{|\partial_d X|}{c_d}$$

over $X \subset V(H^d)$. Interestingly, when all factors of H^d have equal size, the lexicographic and reverse–lexicographic orders both result in structurally symmetric subsets and hence are equivalent with respect to extremal sets minimizing the boundary (dimension–normalized or otherwise). Thus our result is identical to Lindsey's for $k_1 = k_2 = \cdots = k_d$.

We formulated the problem for the 2-dimensional case as the maximization of the function f defined on partitions $\lambda \in \mathbb{P}(m, n)$, $(m \leq n)$, by

$$f(\lambda) = \gamma_n \sum_{i=1}^{n} \lambda_i^2 + \gamma_m \sum_{j=1}^{m} \lambda_j'^2$$

and proved that f is maximized for $N \leq nm$, by the reverse–lexicographically smallest partition of N in $\mathbb{P}(m,n)$, where

$$\gamma_n = \begin{cases} n^2 & if\ n\ is\ even, \\ n^2 - 1 & if\ n\ is\ odd \end{cases}$$

This result for $d = 2$ forms the base step of the higher dimensional case.

In fact it would be interesting to see a characterization of functions θ_n for which

$$f(\lambda) = \frac{\theta_n}{\theta_m} \sum_{i=1}^{n} \lambda_i^2 + \sum_{j=1}^{m} \lambda_j'^2$$

is maximized by the reverse–lexicographically smallest partition of N in $\mathbb{P}(m,n)$.

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