Problem 4: Vector Calculus (20')

Suppose x is a 3-d vector.

$$f(x) = |e^{A \cdot x + b} - c|_2^2$$

where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1.5 & -2 \end{bmatrix}, b = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, c = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

$|\cdot|_2$ is 2-norm: $|x|_2 = \sqrt{x_1^2 + x_2^2 + \ldots}$.

What is the differential $\frac{\partial f}{\partial x}$?

$$\underbrace{x}_{(3\times1)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

dim of $Ax+b$: 2×1 $\rightarrow$ dim of u : 2×1

u^T : 1×2

let $u = e^{Ax+b} - C$

$f(x) = |u|_2^2 = u^T u$

$\frac{df}{du} = \frac{d}{du} u^T u = 2u^T$

$\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}$

$\frac{du}{dx} = \begin{bmatrix} \frac{du_1}{dx_1} & \frac{du_1}{dx_2} & \frac{du_1}{dx_3} \\ \frac{du_2}{dx_1} & \frac{du_2}{dx_2} & \frac{du_2}{dx_3} \end{bmatrix}$

$= \ldots$ simplify

$= 2u^T \frac{du}{dx}$

$$\underbrace{\underset{(1\times2)}{2u^T} \; \underset{(2\times3)}{\frac{du}{dx}}}_{(1\times3)}$$

$$y = x_1^2 + 2x_2^2$$

$$\frac{dy}{dx} = \left[\frac{dy}{dx_1} , \frac{dy}{dx_2} \right]$$

$$= \left[\frac{d(x_1^2 + 2x_2^2)}{dx_1} , \frac{d(x_1^2 + 2x_2^2)}{dx_2} \right]$$

$$= \left[2x_1 , 4x_2 \right]$$

$$y = Ax$$

$$A = \begin{array}{cc} & n \\ m & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad x = \begin{array}{cc} & 1 \\ n & \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{array}$$

$$\frac{dy}{dx} = ? \qquad \underset{(m \times 1)}{y} = Ax = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \frac{dy_i}{dx} = \left[\frac{dy_i}{dx_1}, \frac{dy_i}{dx_2}, \ldots, \frac{dy_i}{dx_n} \right]$$

$$\frac{dy_i}{dx_i} = a_{ij} \qquad \frac{dy}{dx} = \frac{d}{dx} Ax = A$$

$y = x^T A$

$\frac{dy}{dx} = ?$

$x^T = [x_1 \cdots x_m]$

$\downarrow$

$(1 \times m)$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix}$

$\downarrow$

$(m \times n)$

$$y = x^T A = [a_{11}x_1 + \cdots + a_{m1}x_m, \cdots, a_{1n}x_1 + \cdots + a_{mn}x_m]$$

$\downarrow$

$(1 \times n)$

$$y_i = \sum_{j=1}^{m} a_{ji} x_j \qquad \frac{dy_i}{dx_j} = a_{ji}$$

$$\frac{dy}{dx} = \frac{d}{dx} x^T A = A^T$$

$$y = u^T v \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\frac{dy}{dx} = ?$$

$$y = u^T v = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + \cdots + u_m v_m$$

$$= \sum_{i=1}^{m} u_i v_i$$

$$\frac{dy}{dx} = \frac{d}{dx} \sum_{i=1}^{m} u_i v_i$$

product rule

$$\frac{dy}{dx_k} = \sum_{i=1}^{n} \left(u_i \frac{dv_i}{dx_k} + v_i \frac{du_i}{dx_k} \right)$$

$$= \underbrace{\begin{bmatrix} u_1, & \cdots, & u_n \end{bmatrix}}_{u^T} \underbrace{\begin{bmatrix} \frac{dv_1}{dx_k} \\ \vdots \\ \frac{dv_n}{dx_k} \end{bmatrix}}_{\frac{dv}{dx_k}} + \underbrace{\begin{bmatrix} v_1, & \cdots, & v_n \end{bmatrix}}_{v^T} \underbrace{\begin{bmatrix} \frac{du_1}{dx_k} \\ \vdots \\ \frac{du_n}{dx_k} \end{bmatrix}}_{\frac{du}{dx_k}}$$

$$= u^T \frac{dv}{dx} + v^T \frac{du}{dx}$$