1 Lifting Transform

(A combination of Pless notes and my own.)

There is a fascinating relationship between Delaunay triangulations, Voronoi diagrams for two-dimensional points the convex hull of a particular set of 3-dimensional points. At first, Delaunay triangulations (or Voronoi diagrams) and convex hulls appear to be quite different structures: the former uses metric properties (distances) while the latter depends on affine properties (collinearity, coplanarity). The connection between the two is through a Lifting Transform, which maps a set of $d$-dim points to a set of objects (points or hyperplanes) in dimension $d + 1$.

We will demonstrate the connection in dimension 2. The basis of the transform is the paraboloid $z = x^2 + y^2$. This equation defines a surface whose vertical cross sections (constant $x$ or constant $y$) are parabolas, and whose horizontal cross sections (constant $z$) are circles. For each point in the plane, $(x, y)$, the vertical projection of this point onto this paraboloid is $(x, y, x^2 + y^2)$ in 3 space.

Given a set of points $S$ in the plane, let $S_0$ denote the projection of the points in $S$ onto this paraboloid. Now, consider the lower convex hull of $S_0$. This is the portion of the convex hull of $S_0$ which is visible to a viewer standing at $z = -1$. We claim that if we take the lower convex hull of $S_0$, and project it back onto the plane, then we get the Delaunay triangulation...
of $S$. In particular, let $(p, q, r)$ be elements of $S$, and let $p_0, q_0, r_0$ denote the projections of these points onto the paraboloid. Then $p_0q_0r_0$ define a face of the lower convex hull of $S_0$ if and only if $pqr$ is a triangle of the Delaunay triangulation of $S$. The process is illustrated in the following figure.

\section{Delaunay Triangulation}

![Diagram showing the process of projecting points onto a paraboloid, computing the convex hull, and projecting back to the plane.]

The question is, why does this work? To see why, we need to establish the connection between the triangles of the Delaunay triangulation and the faces of the convex hull of transformed points. In particular, recall that

- \textbf{[Delaunay condition:]} Three points $p, q, r$, in $S$ form a Delaunay triangle if and only if the circumcircle of these points contains no other point of $S$.

- \textbf{[Convex hull condition:]} Three points $p_0, q_0, r_0$ in $S_0$ form a face of the convex hull of $S_0$ if and only if the plane passing through $p_0, q_0,$ and $r_0$ has all the points of $S_0$ lying to one side.

Clearly, the connection we need to establish is between the emptiness of circumcircles in the plane and the emptiness of halfspaces in 3 space. We will prove the following claim.

\textbf{Lemma 1.} Consider 4 distinct points $p, q, r, s$ in the plane, and let $p_0, q_0, r_0, s_0$ be their respective projections onto the paraboloid, $z = x^2 + y^2$. The point $s$ lies within the circumcircle of $p, q, r$ if and only if $s_0$ lies on the lower side of the plane passing through $p_0, q_0, r_0$. 


To prove the lemma, first consider an arbitrary (nonvertical) plane in 3
space, which we assume is tangent to the paraboloid above some point \((a, b)\)
in the plane. What is the equation of this tangent plane? We determine the
‘slopes’ of the plane by taking the derivatives of \(z = x^2 + y^2\) with respect
to \(x\) and \(y\), namely, \(dz/dx = 2x\) and \(dz/dy = 2y\), and evaluating them at
the point \((a, b, a^2 + b^2)\). These evaluate to 2a and 2b. Therefore, the plane
passing through these point has the form

\[ z = 2ax + 2by + k \]

To solve for \(k\) we use the fact that the plane passes through \((a, b, a^2 + b^2)\),
and so we can eliminate \(z\) by setting:

\[ a^2 + b^2 = 2a^2 + 2b^2 + k, \]

which gives \(k = -(a^2 + b^2)\). Thus the plane equation is:

\[ z = 2ax + 2by - (a^2 + b^2) \]

Next, if we shift the plane upwards by some positive amount \(\rho^2\) we get
the plane

\[ z = 2ax + 2by - (a^2 + b^2) + \rho^2 \]

How does this plane intersect the paraboloid? Since the paraboloid is
defined by: \(z = x^2 + y^2\), we can eliminate \(z\), giving

\[ x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + \rho^2 \]

which after some simple rearrangements is equal to

\[ (x - a)^2 + (y - b)^2 = \rho^2 \]

This is just a circle. Thus, the intersection of a plane with the paraboloid
produces a space curve (which turns out to be an ellipse), which when pro-
jected back onto the \((x, y)\) coordinate plane is a circle centered at \((a, b)\).
Furthermore, the squared radius of the circle equals the vertical distance be-
tween the projection of the \((a, b)\) onto the paraboloid and its projection onto
the plane.
Thus, we conclude that the intersection of an arbitrary lower halfspace with the paraboloid, when projected onto the \((x, y)\) plane is the interior of a circle. Going back to the lemma, when we project the points \(p, q, r\) onto the paraboloid, the projected points \(p_0, q_0, r_0\) define a plane. Since \(p_0, q_0, r_0\), lie at the intersection of the plane and paraboloid, the original points \(p, q, r\) lie on the projected circle. Thus this circle is the (unique) circumcircle passing through these \(p, q, r\). The point \(s\) lies within this circumcircle, if and only if its projection \(s_0\) onto the paraboloid lies within the lower halfspace of the plane passing through.

![Diagram of geometric relationship between paraboloid and plane]  

3 Voronoi Diagram

Given a point \(p = (a, b)\), the hyperplane \(H(p)\) that is tangent to \(p\)'s lifting, namely, \((a, b, a^2 + b^2)\), has the equation

\[
z = 2ax + 2by - (a^2 + b^2)
\]

Now, consider an arbitrary point \(q = (\alpha, \beta)\) in the plane. What is the vertical distance from \(q\) to the paraboloid? Just \((\alpha^2 + \beta^2)\). What is the vertical distance from \(q\) to plane \(H(p)\)? It is \(2a\alpha + 2b\beta - (a^2 + b^2)\).

Let \(\Delta(p, q)\) denote the difference between these two vertical distance, namely, the additional distance that \(q\)' projection on \(H(p)\) has to travel to reach the paraboloid. We get

\[
\Delta(p, q) = \alpha^2 + \beta^2 - 2(a\alpha + b\beta) + a^2 + b^2 = (a - \alpha)^2 + (b - \beta)^2
\]
That is, $\Delta(p, q)$ equals precisely the two-dimensional distance between $p$ and $q$ in their ambient space.

Now, consider two points $p_1$ and $p_2$ in the plane $z = 0$. We claim that $q$ is closer to $p_1$ if and only if at the position $q = (\alpha, \beta)$, the plane $H(p_1)$ lies above (closer to the paraboloid) than $H(p_2)$. It simply follows from the vertical distance formula. We, therefore, have the following lemma.

**Lemma 2.** Let $p_1, p_2, \ldots, p_n$ be a set of points in the plane $z = 0$. Then, a point $q$ belongs to the Voronoi cell of the point $p_i$ if and only if $H(p_i)$ is the highest plane (seen from $z = +\infty$) at $q$.

Therefore, the Voronoi diagram of $p_1, p_2, \ldots, p_n$ is simply the vertical projection, down to plane $z = 0$, of the point-wise maxima of the downward facings halfspaces $H(p_i)$. Or, equivalently, is the uppermost face of the arrangement defined by these planes.

## 4 Order $k$ Voronoi Diagrams

We can define order $k$ Voronoi diagram as a partition of the plane into convex regions where each region has the same set of $k$ nearest neighbors. The ordinary Voronoi diagram is the order 1. (The relative order of neighbors may change, but the set is the same.)

By the vertical distance argument it is also clear that if we consider the $k$th level in the arrangement formed by the hyperplanes $H(p_i)$, where the topmost level is level 1, then we have the property that for any point on the level $k$, the same $k$ planes lie above (and thus are closest to) the point on the projected space.